Pisot type substitutions

V. Berthé
Laboratoire d'Informatique Algorithmique : Fondements et Applications
LIAFA-UMR 7089
Université Paris Diderot, Paris 7 - Case 7014
F-75205 Paris Cedex 13, France
berthe@liafa.univ-paris-diderot.fr

Abstract

We focus on a class of substitutions, namely the Pisot substitutions. They are defined as follows: Pisot substitutions are primitive substitutions such that the dominant Perron–Frobenius eigenvalue of their incidence matrix is assumed to be a Pisot number, that is, an integer whose conjugates lie strictly inside the unit disk. We discuss their arithmetic and spectral properties. We then extend the corresponding notions to the S-adic framework: one does not consider only the iteration of a single substitution, but infinite compositions of substitutions.

Contents

1 Introduction

One fundamental question concerning symbolic dynamical systems, and in particular, substitutive dynamical systems, deals with the possibility of giving to them a geometric representation. By geometric representation, one considers here dynamical systems of a geometric nature that are measure-theoretically isomorphic to the initial symbolic dynamical system. In particular, one looks for conditions under which it is possible to give a geometric representation of a substitutive dynamical system as a translation on a torus, or on a compact metric group. This latter question can be reformulated in spectral terms: which are the substitutions whose associated dynamical system has discrete spectrum? For more details, see e.g. [51, 49]. Pisot irreducible substitutions, that is, substitutions for which the characteristic polynomial of their incidence matrix is assumed to be the minimal polynomial of a Pisot number, that is, an integer whose conjugates lie strictly inside the unit disk, are assumed to have pure discrete spectrum. This is called the Pisot substitutive conjecture. Pisot substitutive dynamical systems are thus expected to be measure-theoretically isomorphic to a translation on a compact metric group. Note that the conjecture is proved in [12] in the setting of beta-numeration. We focus here on the class of Pisot irreducible substitutions by assuming furthermore that their incidence matrices are unimodular (they have determinant ∓ 1). The translations involved are thus toral translations. We first discuss their arithmetic and spectral properties within the Pisot conjecture framework. We then extend the corresponding notions to the so-called S-adic framework.

More precisely, we recall basic notions on substitutions and symbolic dynamical systems in Section 2. Section 3 is devoted to Pisot substitutions. The notion of discrete spectrum is recalled in Section 4, and Rauzy fractals are discussed in Section 5. Section 6 recalls basic definitions concerning S-adic systems. Section

7 introduces the S-adic counterpart of the notion of Pisot substitution. Lastly Section 8 handles the cases of Arnoux-Rauzy and Brun S-adic systems.

The material of this lecture comes mostly from [51, Chap. 1], [15] and [19]. I warmly thank the coauthors of these papers for using the present material. These publications are available at `http://www.liafa.univ-paris-diderot.fr/~berthe/` and at `http://www.liafa.univ-paris-diderot.fr/~berthe/Fogg.html`.

2 Substitutions and symbolic dynamical systems

Substitutions. We consider a finite set of *letters* $\mathcal{A}$, called *alphabet*. A (finite) *word* is an element of the free monoid $\mathcal{A}^*$ generated by $\mathcal{A}$. A *substitution* σ over the alphabet $\mathcal{A}$ is a non-erasing endomorphism of the free monoid $\mathcal{A}^*$ (non-erasing means that the image of any letter is not equal to the empty word but contains at least one letter).

For $i \in \mathcal{A}$ and for $w \in \mathcal{A}^*$, let $|w|_i$ stand for the number of occurrences of the letter i in the word w. Let d stand for the cardinality of $\mathcal{A}$. Let σ be a substitution. Its *incidence matrix* $M_\sigma = (m_{i,j})_{1\leq i,j\leq d}$ is defined as the square matrix with entries $m_{i,j} = |\sigma(j)|_i$ for all i, j. A substitution is said *primitive* if there exists a power of its incidence matrix whose entries are all positive. We say that σ is *unimodular* if $\det(M_\sigma) = \pm 1$.

The set $\mathcal{A}^{\mathbb{N}}$ shall be equipped with the product topology of the discrete topology on each copy of $\mathcal{A}$. Thus, this set is a compact space. This topology is the topology defined by the following distance:

$$\text{for } u \neq v \in \mathcal{A}^{\mathbb{N}},\ d(u,v) = 2^{-\min\{n\in\mathbb{N};\ u_n\neq v_n\}}.$$

Thus, two infinite words are close to each other if their first terms coincide. Note that the space $\mathcal{A}^{\mathbb{N}}$ is complete as a compact metric space. Furthermore, it is a *Cantor set*, that is, a totally disconnected compact set without isolated points.

A *fixed point* of a substitution σ is an infinite word $u = (u_n)_n$ with $\sigma(u) = u$. A *periodic point* of σ is an infinite word u with $\sigma^k(u) = u$ for some $k > 0$.

Substitutions are very efficient tools for producing infinite words. Let σ be a substitution over the alphabet $\mathcal{A}$, and a be a letter such that $\sigma(a)$ begins with a and $|\sigma(a)| \geq 2$. Then there exists a unique fixed point u of σ beginning with a. This infinite word is obtained as the limit in $\mathcal{A}^\star \cup \mathcal{A}^{\mathbb{N}}$ (when n tends toward infinity) of the sequence of words $(\sigma^n(a))_n$, which is easily seen to converge (the topology on $\mathcal{A}^{\mathbb{N}}$ is extended to $\mathcal{A}^\star \cup \mathcal{A}^{\mathbb{N}}$ by adding an extra symbol to the alphabet $\mathcal{A}$).

Example 1 (Fibonacci substitution)**.** We consider the substitution σ on $\mathcal{A} = \{a, b\}$ defined by $\sigma(a) = ab$ and $\sigma(b) = a$. Its incidence matrix is

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then, the sequence of finite words $(\sigma^n(a))_n$ starts with

$$\sigma^0(a) = a,\ \sigma^1(a) = ab,\ \sigma^2(a) = aba,\ \sigma^3(a) = abaababa,\ \ldots$$

Each $\sigma^n(a)$ is a prefix of $\sigma^{n+1}(a)$, and the limit word in $\mathcal{A}^{\mathbb{N}}$ is

$$abaababaabaababaababaabaababaabaababaababaabaababaababaabaab\ldots$$

The above limit word is called the *Fibonacci word* (for more on the Fibonacci word, see e.g. [48, 51, 55]).

Primitivity. An infinite word $u = (u_n)_n$ is *uniformly recurrent* if every word occurring in u occurs in an infinite number of positions with bounded gaps, that is, if for every factor w, there exists s such that for every n, w is a factor of $u_n \ldots u_{n+s-1}$. The set of factors $\mathcal{L}_u$ of an infinite word u is called its *language*.

We recall that if σ is primitive, any of its periodic points is uniformly recurrent. Indeed, let $u = \sigma^p(u)$ $(p \geq 1)$ be a periodic point of σ. We have $u = (\sigma^p)^k(u) = (\sigma^p)^k(u_0)(\sigma^p)^k(u_1)\ldots$; for any $b \in \mathcal{A}$, a occurs in $(\sigma^p)^k(b)$, hence a occurs in u infinitely often with bounded gaps; but then so does every $(\sigma^p)^n(a)$ in $u = (\sigma^p)^n(u)$, hence so does any word occurring in u.

According to Perron–Frobenius' theorem, if a substitution is primitive, then its incidence matrix admits a dominant eigenvalue (it dominates strictly in modulus the other eigenvalues) that is (strictly) positive. It is called its *Perron–Frobenius eigenvalue*, or else its *expansion* factor.

Symbolic dynamical system. Let S stand for the (one-sided) *shift* acting on $\mathcal{A}^{\mathbb{N}}$:

$$S((u_n)_{n\in\mathbb{N}}) = (u_{n+1})_{n\in\mathbb{N}}.$$

One can associate with any infinite word in $\mathcal{A}^{\mathbb{N}}$ a symbolic dynamical system, defined as a closed shift invariant subset of $\mathcal{A}^{\mathbb{N}}$. Indeed, let $u \in \mathcal{A}^{\mathbb{N}}$. Let $\overline{\mathcal{O}(u)}$ be the positive orbit closure of the infinite word u under the action of the shift S, i.e., the closure in $\mathcal{A}^{\mathbb{N}}$ of the set $\mathcal{O}(u) = \{S^n(u) \mid n \geq 0\}$. The *substitutive symbolic dynamical system* (X_u, S) (also called *shift*) generated by u is defined as $X_u := \overline{\mathcal{O}(u)}$. Its set of factors is called its *language*.

We can also associate such a symbolic system with a primitive substitution. Let σ be a primitive substitution over $\mathcal{A}$. Let $u \in \mathcal{A}^{\mathbb{N}}$ be such that $\sigma^k(u) = u$ for some $k \geq 1$. Such an infinite word exists by primitivity of σ. Indeed, there exist a letter a and a positive integer k such that $\sigma^k(a)$ begins with a; consider as first letter of u this letter a; take $u = \lim_{n\to\infty} \sigma^{kn}(a)$. Let again $\overline{\mathcal{O}(u)}$ be the positive orbit closure of the infinite word u under the action of the shift S, i.e., the closure of the set $\mathcal{O}(u) = \{S^n(u) \mid n \geq 0\}$. The substitutive symbolic dynamical system (X_σ, S) generated by σ is defined as $X_\sigma := \overline{\mathcal{O}(u)}$. One easily checks by primitivity that (X_σ, S) does not depend on the choice of the infinite word u fixed by some power of σ. For more details, see e.g. [49].

The dynamical system (X_σ, S) associated with a primitive substitution σ can be endowed with a Borel probability measure μ *invariant* under the action of the shift S, that is, $\mu(S^{-1}B) = \mu(B)$, for every Borel set B. Indeed, this measure is uniquely defined by its values on the cylinders. The *frequency* of a letter i in an infinite word u is defined as the limit when n tends towards infinity, if it exists, of the number of occurrences of i in $u_0u_1 \cdots u_{n-1}$ divided by n. For a given (finite) word w of the language of X_σ, the cylinder $[w]$ is the set of infinite words in X_σ that have w as a prefix. The measure of the cylinder $[w]$ is thus defined as the frequency of the finite word w in any element of X_σ, which does exist (σ is assumed to be primitive). For more details, see the lecture [34] or see the book [49]. Let us recall also that the system (X_σ, S) is *uniquely ergodic* if σ is assumed to be primitive: there exists a unique shift-invariant measure.

More generally, let us now recall the main properties of the symbolic systems (X_σ, S) associated with primitive substitutions. For more details, see [34] or [49]. The (*factor*) *complexity function* of an infinite word u counts the number of distinct factors of a given length. An infinite word u is said to be *linearly recurrent* if there exists a constant C such that $R(n) \leq Cn$, for all n.

Theorem 1. *Let σ be a primitive substitution. Then, (X_σ, S) is minimal, linearly recurrent, uniquely ergodic. Any of its elements has at most linear factor complexity.*

Note that all there results hold also for biinfinite words in $\mathcal{A}^{\mathbb{Z}}$. In this case, the shift S is invertible.

For analogue notions of substitutions and associated dynamical systems defined on tilings and point sets, and acting as inflation/subdivision rules, see the surveys [61, 56, 50].

3 First properties of Pisot substitutions

We now concentrate on Pisot substitutions.

3.1 Pisot substitutions

Let us recall that an algebraic integer $\alpha > 1$ is a *Pisot-Vijayaraghavan number* or a *Pisot number* if all its algebraic conjugates λ other than α itself satisfy $|\lambda| < 1$. This class of numbers has been intensively studied and has some special Diophantine properties (see for instance [30]).

Example 2. The largest roots of $X^2 - X - 1$, $X^3 - X^2 - X - 1$, $X^3 - X - 1$ or else $X^3 - X^2 - 1$ are Pisot numbers.

A primitive substitution is said to be *Pisot* if its expansion number (i.e., its Perron–Frobenius eingenvalue) is a Pisot number.

A primitive substitution is said *Pisot irreducible* if the characteristic polynomial of its incidence matrix is the minimal polynomial of a Pisot number.

Example 3. The Fibonacci substitution is a Pisot irreductible substitution.

3.2 Balancedness properties

Frequencies and invariant measures. Let u be an infinite word. Recall that the *frequency* of a letter i in u is defined as the limit when n tends towards infinity, if it exists, of the number of occurrences of i in $u_0 u_1 \cdots u_{n-1}$ divided by n. The vector f whose components are given by the frequencies of the letters is called the *letter frequency vector*. The infinite word u has *uniform letter frequencies* if, for every letter i of u, the number of occurrences of i in $u_k \cdots u_{k+n-1}$ divided by n has a limit when n tends to infinity, *uniformly in k*.

Similarly, we can define the frequency and the uniform frequency of a factor, and we say that u has *uniform frequencies* if all its factors have uniform frequency.

Note that having frequencies is a property of an infinite word u while having uniform frequencies is a property of the associated language $\mathcal{L}_u$ or of the shift X_u. Recall that a probability measure μ on X_u is said invariant if $\mu(S^{-1}A) = \mu(A)$ for every measurable set $A \subset X_u$. An invariant probability measure on X_u is *ergodic* if any shift-invariant measurable set has either measure 0 or 1. If μ is an ergodic measure on u, then we know from the Birkhoff ergodic Theorem that μ-almost every infinite word in X_u has frequency $\mu([w])$, for any cylinder[1] $[w]$, but this frequency is not necessarily uniform. If the shift X_u is uniquely ergodic (i.e., there exists a unique shift-invariant probability measure on X_u), then the unique invariant measure on X is ergodic and the convergence in the Birkhoff ergodic Theorem holds uniformly for every infinite word in X_u. The property of having uniform factor frequencies for a shift is actually equivalent to unique ergodicity. In that case, one recovers the frequency of a factor w of length n as $\mu([w])$. For more details on invariant measures and ergodicity, we refer to [49] and [16, Chap. 7].

Discrepancy and balancedness. Let $u \in \mathcal{A}^{\mathbb{N}}$ be an infinite word and assume that each letter i has frequency f_i in u. The *discrepancy* of u is

$$\Delta(u) = \limsup_{i \in \mathcal{A},\ n \in \mathbb{N}} ||u_0 u_1 \ldots u_{n-1}|_i - n f_i|.$$

The quantity $\Delta(u)$ is considered e.g. in [1, 2]. We also consider

$$\Delta_n(u) = \sup_{i \in \mathcal{A}} ||u_0 u_1 \ldots u_{n-1}|_i - n f_i|.$$

An infinite word $u \in \mathcal{A}^{\mathbb{N}}$ is said to be *C-balanced* if for any pair v, w of factors of the same length of u, and for any letter $i \in \mathcal{A}$, one has $||v|_i - |w|_i| \leq C$. It is said *balanced* if there exists $C > 0$ such that it is C-balanced.

Proposition 1. *An infinite word $u \in \mathcal{A}^{\mathbb{N}}$ is balanced if and only if it has uniform letter frequencies and there exists a constant B such that for any factor w of u, we have $||w|_i - f_i|w|| \leq B$ for all letter i in $\mathcal{A}$, where f_i is the frequency of i. Moreover, if u has letter frequencies, then u is balanced if and only if its discrepancy $\Delta(u)$ is finite.*

[1] Recall that $[w] = \{v \in X_u ; v_0 \ldots v_{n-1} = w\}$.

Proof. We follow the proof given in [15]. Let u be an infinite word with letter frequency vector f and such that $||w|_i - f_i|w|| \leq B$ for every factor w and every letter i in $\mathcal{A}$. Then, for every pair of factors w_1 and w_2 with the same length n, we have by triangular inequality

$$||w_1|_i - |w_2|_i| \leq ||w_1|_i - nf_i| + ||w_2|_i - nf_i| \leq 2B.$$

Hence L is $2B$-balanced (see also [1, Proposition 7]).

Conversely, assume that u is C-balanced for some $C > 0$. We fix a letter $i \in \mathcal{A}$. For every non-negative integer p, let N_p be defined as an integer N such that for every word of length p of u, the number of occurrences of the letter i belongs to the set $\{N, N+1, \cdots, N+C\}$.

We first observe that the sequence $(N_p/p)_{p\in\mathbb{N}}$ is a Cauchy sequence. Indeed consider a factor w of length pq, where $p, q \in \mathbb{N}$. The number $|w|_i$ of occurrences of i in w satisfies

$$pN_q \leq |w|_i \leq pN_q + pC, \quad qN_p \leq |w|_i \leq qN_p + qC.$$

We deduce that $-qC \leq qN_p - pN_q \leq pC$ and thus $-C \leq N_p - pN_q/q \leq pC/q$.

Let f_i stand for $\lim_q N_q/q$. By letting q tend to infinity, one then deduces that $-C \leq N_p - pf_i \leq 0$. Thus, for any factor w of u we have

$$\left|\frac{|w|_i}{|w|} - f_i\right| \leq \frac{C}{|w|},$$

which was to be proved.

If u has letter frequencies, the equivalence between balancedness and finite discrepancy comes from triangular inequality. □

Sturmian words (see Section 6.2) are known to be 1-balanced [48]; they even are exactly the 1-balanced infinite words that are not eventually periodic. There exist Arnoux-Rauzy words (see Section 6.3) that are not balanced such as first proved in [28], contradicting the belief that they would be natural codings of toral translations. For more on this subject, see also [18, 29].

We follow the conventions of [1]. Let σ be a primitive substitution and λ be its Perron-Frobenius eigenvalue. Let d' stand for the number of distinct eigenvalues of M_σ. Let λ_i, for $i = 1, \cdots, d'$, stand for the eigenvalues of σ, with $\lambda_1 = \lambda$, and let $\alpha_i + 1$ stand for their multiplicities in the minimal polynomial of the incidence matrix M_σ. We order them as follows. Let i, k such that $2 \leq i < k \leq d'$. If $|\lambda_i| \neq |\lambda_k|$, then $|\lambda_i| > |\lambda_k|$. If $|\lambda_i| = |\lambda_k|$, then $\alpha_i \geq \alpha_k$. We also add that if $|\lambda_i| = |\lambda_k| = 1$, and $\alpha_i = \alpha_k$, if λ_i is not a root of unity and λ_k is a root of unity, then $i < k$. Note that several orders satisfy these conditions but this will cause no problem since the results described in this section do not depend on the choice of such an order (see [1, Remark 1]).

Theorem 2. *Primitive Pisot substitutions are balanced, and have finite discrepancy.*

Proof. The proof follows the proof of [2, Proposition 11] and uses the Dumont-Thomas prefix-suffix numeration [32].

Let σ be a primitive Pisot substitution over the alphabet $\mathcal{A}$. Let us prove that σ has finite discrepancy. Let $(f_i)_i$ stand for its letter frequency vector. We consider the abelianization map l defined as the map

$$l\colon \mathcal{A}^* \to \mathbb{N}^d,\ w \mapsto (|w|_1, |w|_2, \cdots, |w|_d).$$

Note that

$$l(\sigma(w)) = M_\sigma l(w),$$

for any word w.

We first consider a fixed word w of the form $w = \sigma^n(j)$, for j letter in $\mathcal{A}$. If i is a fixed letter in $\mathcal{A}$, the sequence $(|\sigma^n(j)|_i)_n$ satisfies a linear recurrence whose coefficients are provided by the minimal polynomial of M_σ. Hence, there exists $C_{i,j}$ such that

$$|\sigma^n(j)|_i = C_{i,j} f_i \lambda^n + O(n^{\alpha_2}|\lambda_2|^n).$$

By applying the Perron–Frobenius Theorem, one checks that there exists C_j such that $C_{i,j} = C_j f_i$ for all i, hence

$$|\sigma^n(j)|_i = C_j f_i \lambda^n + O(n^{\alpha_2}|\lambda_2|^n).$$

We then deduce from $\sum_i f_i = 1$ that

$$|\sigma^n(j)|_i - f_i|\sigma^n(j)| = O(n^{\alpha_2}|\lambda_2|^n).$$

It remains to check that this result also holds for prefixes of the fixed point u. Indeed, it is easy to prove that any prefix w of u can be expanded as:

$$w = \sigma^k(w_k)\sigma^{k-1}(w_{k-1})\ldots w_0,$$

where the w_i belong to a finite set of words. (This corresponds to a "numeration system" on words; there are some admissibility conditions on the possible sequences (w_i), which can be worked out explicitly: they are given by a finite automaton.) This numeration is called Dumont-Thomas numeration. □

In fact, more can be said concerning balance properties of primitive substitutions. We follow the convention introduced previously on the spectrum of the incidence matrix M_σ.

Theorem 3 ([1, 2])**.** *Let σ be a primitive substitution. Let u be a fixed point of σ.*

- *If $|\lambda_2| < 1$, then the discrepancy $\Delta(u)$ is finite..*
- *If $|\lambda_2| > 1$, then $\Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2} n^{(\log_\lambda |\lambda_2|)})$.*

- *If $|\lambda_2| = 1$, and λ_2 is not a root of unity, then*

$$\Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2+1}).$$

If λ_2 is a root of unity, then either

$$\Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2+1}), \ \textit{or}\ \Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2}).$$

In particular there exist balanced fixed points of substitutions for which $|\theta_2| = 1$. All eigenvalues of modulus one of the incidence matrix have to be roots of unity.

Observe that the Thue-Morse word is 2-balanced, but if one considers generalized balances with respect to factors of length 2 instead of letters, then it is not balanced anymore.

3.3 Birkhoff sums

Let $u = (u_n)_n \in \mathcal{A}^{\mathbb{N}}$ be an infinite word and $\phi : \mathcal{A}^{\mathbb{N}} \to \mathbb{R}$ be a continuous function. The *Birkhoff sum* of ϕ along u is the sequence

$$S_n(\phi, u) = \phi(u) + \phi(Su) + \ldots + \phi(S^{n-1}u).$$

It generalizes the concept of frequency: indeed, if $\phi = \mathbf{1}_{[i]}$ is the characteristic function of the letter i, then $S_n(\phi, u)$ is the number of occurrences of i in the prefix of length n of u. More generally, if (X_u, S) is the symbolic dynamical system generated by u, and $\phi : A^{\mathbb{N}} \to \mathbb{R}$ is a continuous function, we may define

$$S_n(\phi, S, v) = \phi(v) + \phi(Sv) + \ldots + \phi(S^{n-1}v),$$

for all $v \in X_u$. In the context of symbolic dynamics, this corresponds to take the sum of the values of ϕ along the orbit of v under the action of the shift S.

Assume that (X_u, S) has uniform word frequencies and let f denote the letter frequencies vector. Then, uniformly in $v \in X_u$, we have, by unique ergodicity,

$$\lim_{n\to\infty} \frac{S_n(\phi, S, v)}{n} = \sum_{i \in A} \phi(i) f_i.$$

Now, u is balanced if and only if there exists a constant C so that

$$\left| \frac{S_n(\phi, T, x)}{n} - \sum_{i \in A} \phi(i) f_i \right| \leq \frac{C\|\phi\|}{n}, \quad \text{for all } n \geq 0.$$

In other words, balancedness may be interpreted as an optimal speed of convergence of Birkhoff sums.

4 Group translations and discrete spectrum

We recall in this section elements concerning the notion of pure discrete spectrum for symbolic dynamical systems.

4.1 Spectrum

We first recall some basic definitions concerning the spectrum of a dynamical system. Good references on the subject are [31, 64].

Note that we consider here invertible measure-theoretic dynamical systems (X, T, μ), with T is *invertible*, and T^{-1} being also measurable and measure-preserving. If one works with infinite words (and not with biinfinite words), the shift S is a priori not invertible. Nevertheless, all the previous setting (introduced in Section 2) extends in a straightforward way to biinfinite words in $\mathcal{A}^{\mathbb{Z}}$.

Let (X, T, μ) be an invertible measure-theoretic dynamical system. One can associate with it in a natural way an *operator* U acting on the Hilbert space $\mathcal{L}^2(X, \mu)$ defined as the following map:

$$\begin{array}{rrcl} U: & \mathcal{L}^2(X,\mu) & \to & \mathcal{L}^2(X,\mu) \\ & f & \mapsto & f \circ T. \end{array}$$

This operator is called the *Koopman operator*. Since T preserves the measure, the operator U is easily seen to be a *unitary operator*. Note that the surjectivity of the operator U comes from the invertibility of the map T.

The *eigenvalues* of (X, T, μ) are defined as being those of the map U. The set of eigenvalues of the operator U is called *spectrum*. It is a subgroup of the unit circle. The *eigenfunctions* of (X, T, μ) are defined to be the eigenvectors of U. Let us note that the map U always has 1 as an eigenvalue and any non-zero constant function is a corresponding eigenfunction.

One can deduce ergodic information from the spectral study of the operator U. In particular, T is ergodic if and only if the eigenvalue one is a simple eigenvalue, that is, if all eigenfunctions associated with 1 are constant almost everywhere. Indeed, if a Borel set E of non-trivial measure satisfies $T^{-1}E = E$, then the characteristic function $\mathbf{1}_E$ is a non-constant eigenfunction associated with one. Recall that if the system (X, T, μ) is ergodic, then every Borel function which is T-invariant is almost everywhere constant. Otherwise, if f is not constant almost everywhere, then one can cut its image into two disjoint sets, whose inverse images have a non-trivial measure and are invariant sets.

Furthermore, if T is ergodic, every eigenfunction is simple and every eigenfunction is of constant modulus. Indeed, if f is an eigenfunction for the eigenvalue β, $|f|$ is an eigenfunction for the eigenvalue $|\beta| = 1$ and hence is a constant. If f_1 and f_2 are eigenfunctions for β, $|f_2|$ is a non-zero constant, and f_1/f_2 is an eigenfunction for 1 and hence a constant.

The spectrum is said to be *discrete* (or to have *pure point spectrum*) if $\mathcal{L}^2(X, \mu)$ admits an Hilbert basis of eigenfunctions, that is, if the eigenfunctions span $\mathcal{L}^2(X, \mu)$. Hence, if $\mathcal{L}^2(X, \mu)$ is separable (this is the case for instance if X is a compact metric set), then there are at most a countable number of eigenvalues.

If the spectrum contains only the eigenvalue 1, with multiplicity 1, the system is said to be *weakly mixing* or to have a *continuous spectrum*. This implies in particular that T is ergodic.

4.2 Group translations

The simplest examples of dynamical systems are *toral translations*, and more generally, *translations over a compact group G*, the invariant probability measure being the Haar measure. These are also called *rotations* of the group G. We focus here (in the context of unimodular substitutions) on *toral translations*, that is, on translations R_α by the vector $\alpha = (\alpha_1, \dots, \alpha_d)$ on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$:

$$R_\alpha(x_1, \cdots, x_d) = (x_1 + \alpha_1, \cdots, x_d + \alpha_d) \text{ modulo } 1.$$

More generally, the class of translations over compact groups contains in particular all the toral translations, the additions over a finite group, translations over p-adic integer groups, or else, translations over p-adic solenoids.

One has the following equivalence (for a proof, see for instance [64]).

Theorem 4. *Let G be a compact metric group, and let $T : G \to G$, $x \mapsto ax$ be a translation of G. The following properties are equivalent:*

- *T is minimal;*
- *T is ergodic;*
- *T is uniquely ergodic;*
- *$\{a^n;\ n \in \mathbb{N}\}$ is dense in G.*

We thus deduce by using the density of $\{a^n;\ n \in \mathbb{N}\}$ that the ergodicity of T implies that G is abelian. In particular, a rotation with irrational angle on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is minimal and uniquely ergodic, the invariant measure being the Haar measure. More generally, following Kronecker's theorem, the minimality of toral translations can thus be expressed as follows (for a proof, see for instance [45]).

Proposition 2. *The translation by the vector $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d$ on the d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is minimal if and only if $\alpha_1, \dots, \alpha_d$ and 1 are rationally independent.*

Group translations are known to have discrete spectrum, but one has even more: systems with discrete spectrum are in fact group translations. This is detailed in the two next theorems.

Theorem 5 (Spectrum of group translations)**.** *Any group translation (G, T, μ) (with G compact abelian group G equipped with the Haar measure μ) has discrete spectrum. In particular, the spectrum of the rotation R_α on the one-dimensional torus $\mathbb{T}$ with irrational angle α is the group* $\exp(2i\pi\mathbb{Z}\alpha) = \{e^{2i\pi\, k\alpha};\ k \in \mathbb{Z}\}$. *Similarly, the spectrum of the translation of angle $(\alpha_1, \dots \alpha_d)$ on the d-dimensional torus $\mathbb{T}^d$ is the group* $\exp(2i\pi \sum_j \mathbb{Z}\alpha_j) = \{e^{2i\pi\, \sum_j k_j\alpha_j};\ k_j \in \mathbb{Z}\}$.

The following statement, that can be considered as a reciprocal of the previous result, is due to [63]. See also [64] and see [54] for a nice historical account.

Theorem 6. *[63] Any invertible and ergodic system with discrete spectrum is measure-theoretically isomorphic to a translation on a compact abelian group, equiped with the Haar measure.*

The proof of this assertion is based on the following idea: consider the group Λ of eigenvalues of the operator U endowed with the discrete topology; the group of the translation will be the character group of Λ, which is compact and abelian.

Note that this connection between discrete spectrum and group translations also holds in the topological setting. Indeed, group translations are topological dynamical systems and Theorem 6 has its counterpart in topological terms.

We first introduce the corresponding definitions for topological spectrum. Let (X, T) be a topological dynamical system, where T is an homeomorphism. A non-zero complex-valued continuous in $\mathcal{C}(X)$ is an *eigenfunction* for T if there exists $\lambda \in \mathbb{C}$ such that $\forall x \in X,\ f(Tx) = \lambda f(x)$. The set of the eigenvalues corresponding to those eigenfunctions is called the *topological spectrum* of the operator U. Recall that two dynamical systems (X, S) and (Y, T) are said to be *topologically conjugate* (or *topologically isomorphic*) if there exists an homeomorphism f from X onto Y which conjugates S and T, that is, $f \circ S = T \circ f$. If two systems are topologically conjugate, then they have the same group of eigenvalues. The operator U is said to have *topological discrete spectrum* if the eigenfunctions span $\mathcal{C}(X)$.

Example 4. An ergodic translation on a compact metric abelian group has topological discrete spectrum.

The topological version of theorem of Von Neumann becomes (see for instance [64]):

Theorem 7. *Any invertible and minimal topological dynamical system with topological discrete spectrum is topologically conjugate to a minimal translation on a compact abelian group.*

4.3 Factors of substitutive dynamical systems

In this section, we illustrate the important connection between the eigenvalues of a measure-theoretical dynamical system and its translation factors.

Two measure-theoretic dynamical systems $(X_1, T_1, \mu_1, \mathcal{B}_1)$ and $(X_2, T_2, \mu_2, \mathcal{B}_2)$ are said to be *measure-theoretically isomorphic* if there exist two sets of full measure $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$, a measurable map $f : B_1 \to B_2$ such that

- the map f is one-to-one,
- the reciprocal map of f is measurable,
- f conjugates T_1 and T_2 over B_1 and B_2,
- μ_2 is the image $f_*\mu_1$ of the measure μ_1 with respect to f, that is,

$$\forall B \in \mathcal{B}_2,\ \mu_1(f^{-1}(B)) = \mu_2(B).$$

If the map is f is only onto, then $(X_2, T_2, \mu_2, \mathcal{B}_2)$ is said to be a *measure-theoretic factor* of $(X_1, T_1, \mu_1, \mathcal{B}_1)$.

Lemma 1. *The spectrum of a measure-theoretic dynamical system contains the spectrum of any of its measure-theoretic factors.*

Proof. Let (X_1, T_1, μ_1) be a factor of (X, T, μ). Let f be the conjugacy map. Let g_1 be an eigenfunction of (X_1, T_1, μ_1) for the eigenvalue λ. We have $g_1 \circ T_1 = \lambda\, g_1$. Let $g = g_1 \circ f$. Then $g \circ T = g_1 \circ f \circ T = g_1 \circ T_1 \circ f = \lambda\, g_1 \circ f = \lambda\, g$. Thus, g is an eigenfunction of (X, T, μ) for λ. □

In the other direction, the following lemma states that the knowledge on the existence of some eigenvalue of a dynamical system allows the determination of some translation factor.

Lemma 2. *A rotation R_α of irrational angle α on the one-dimensional torus $\mathbb{T}$ is a measure-theoretic factor of an ergodic dynamical system (X, T, μ) if and only if $e^{2i\pi\,\alpha}$ is an eigenvalue of (X, T, μ); its spectrum then contains* $\exp(2i\pi\mathbb{Z}\alpha)$.

Proof. The necessary condition is a consequence of Lemma 1.

Let g be an eigenfunction of (X, T, μ) for the eigenvalue $e^{2i\pi\,\alpha}$. Let $\mathbb{U} := \{z \in \mathbb{C},\ |z| = 1\}$. We will prove that the rotation $T_\alpha : \mathbb{U} \to \mathbb{U},\ x \mapsto e^{2i\pi\,\alpha}x$ is a measure-theoretic factor of (X, T, μ). This is equivalent (since $(\mathbb{U}, T_\alpha)$ and $(\mathbb{T}, R_\alpha)$ are conjugate) to the fact that $(\mathbb{T}, R_\alpha)$ is a factor of (X, T, μ).

By ergodicity, g is of constant modulus, which can be chosen equal to 1. We thus have $g : X \to \mathbb{U}$, with $g \circ T = e^{2i\pi\,\alpha}g = T_\alpha \circ g$. It remains to prove that g is onto.

We have $\mu(g^{-1}\mathbb{U}) = \mu(X) = 1 \neq 0$, and the measure $\mu_* g$ (i.e., $\mu(g^{-1}(\,))$) on $\mathbb{U}$ is non-zero and invariant under T_α. By unique ergodicity of T_α, this measure is nothing else than the Haar measure. Since $g(X)$ is invariant under T_α and of non-zero measure, we get $g(X) = \mathbb{U}$, by ergodicity of T_α, and $(\mathbb{U}, T_\alpha)$ is a measure-theoretic factor of (X, T, μ). □

Remark 1. More generally, a minimal translation R_α on the torus $\mathbb{T}^d$ is a measure-theoretic factor of an ergodic dynamical system (X, T, μ) if and only if, for every $1 \leq i \leq d$, $e^{2i\pi\alpha_i}$ is an eigenvalue of (X, T, μ) (with $\alpha = (\alpha_1, \ldots, \alpha_d)$); its spectrum then contains $\exp(2i\pi \sum_j \mathbb{Z}\alpha_j)$.

One interesting point in the spectral study of substitutive dynamical systems is that we do not need to distinguish between topological and measure-theoretic eigenfunctions, and thus between topological and measure-theoretic factors.

Theorem 8 (B. Host [42])**.** *Let σ be a primitive and not shift-periodic substitution. Then, any class (in $\mathcal{L}^2$) of eigenfunctions contains a continuous eigenfunction.*

4.4 Bounded remainder sets

We first start with a notion issued from classical discrepancy theory.

A subset A of $\mathbb{T}^d$ with (Lebesgue) measure $\mu(A)$ is said to be a *bounded remainder set* for the translation $R_\alpha : x \mapsto x + \alpha$ ($\alpha \in \mathbb{T}^d$) if there exists $C > 0$ such that for a.e. x the following holds:

$$\forall N, \quad |\mathrm{Card}\{0 \leq n < N\,;\, x + n\alpha \in A\} - N\mu(A)| \leq C.$$

Let $f := \mathbf{1}_A(x) - \mu(A)\mathbf{1}$. The notation $\mathbf{1}$ stands for the constant function that takes value 1. Note that

$$\mathrm{Card}\{n < N\,;\, R_\alpha^n(x) \in A\} - N\mu(A) = \sum_{n<N} f(R_\alpha^n x).$$

Hence, A is a bounded remainder set if and only if the Birkhoff sum $\sum_{n<N} f(R_\alpha^n x)$ is a.e. uniformly bounded.

In the symbolic setting, if (X_u, S) is a uniquely ergodic symbolic shift, a cylinder $[w]$ is called a *bounded remainder set* if the following quantity $\Delta_{[w]}(u)$ is bounded.

$$\Delta_{[w]}(u) = \limsup_{n\in\mathbb{N}} ||u_0 u_1 \ldots u_{n-1}|_w - nf_w|.$$

Here the notation $|v|_w$ stands for the number of occurrences of the word w in v, and f_w stands for the frequency of the word w.

We now recall a classical statement in topological dynamics: bounded deviations yield the existence of a coboundary. This will allow us to exhibit eigenfunctions. For more on the connections between this statement and bounded remainder sets, see the survey [40].

Theorem 9 (Gottschalk–Hedlund [39])**.** *Let X be a compact metric space and $T\colon X \to X$ be a minimal homeomorphism. Let $f\colon X \to \mathbb{R}$ be a continuous function. Then*

$$f = g - g \circ T$$

for a continuous function g if and only if there exists $C > 0$ such that

$$|\sum_{n=0}^{N} f(T^n(x))| < C$$

for all N and all x.

Proposition 3. *Let σ be a primitive substitution. Let w be a finite factor of its language. If the cylinder $[w]$ is a bounded remainder set, then its frequency f_w is an eigenvalue of (X_σ, S).*

Proof. Assume that the cylinder $[w]$ is a bounded remainder set. Let $f = \mathbf{1}_{[w]}(x) - f_w\mathbf{1}$. Since $[w]$ is a bounded remainder set, we can apply Theorem 9: there exists g such that $f = g - g \circ S$.

Note that $e^{2i\pi \mathbf{1}_{[w]}(v)} = 1$ for any $v \in X_\sigma$. This yields

$$\exp^{2i\pi g \circ S} = \exp^{2i\pi f_w} \exp^{2i\pi g}.$$

Hence $\exp^{2i\pi g}$ is an eigenfunction of the operator U associated with the eigenvalue f_w. □

Proposition 4. *Let σ be a primitive Pisot substitution. Then (X_σ, S) admits a toral translation factor: let α denote the dominant eigenvalue of the incidence matrix of σ; the spectrum of the substitutive dynamical system associated with σ contains the set* $\exp(2\pi i\mathbb{Z}[\alpha])$. *In particular, substitutive dynamical systems of Pisot type are never weakly mixing. Furthermore any irreducible Pisot substitutive dynamical system over d letters admits as a factor a minimal translation on the torus $\mathbb{T}^{d-1}$.*

Proof. The proof relies on Lemma 2, Remark 1 and on the fact that $\mathbb{Z}[\alpha]$ is of rank $d-1$ if σ is assumed to be irreducible. Indeed, one checks that the coordinates of the letter frequency vector f (that is, the right eigenvector of M_σ associated with α normalized so that the sum of its coordinates is equal to one) belong to $\mathbb{Q}(\alpha)$. These coordinates are in fact rationally independent. Indeed, consider a non-trivial linear relation $\sum_{i=1}^d f_i r_i = 0$ for some vector $r = (r_i)$ with integer entries. We then use the fact that the eigenvalues of the incidence matrix are simple and algebraic conjugates; the corresponding eigenvectors obtained by applying the canonical Galois automorphisms to the vector $f = (f_i)$ with coordinates in $\mathbb{Q}(\alpha)$ (one replaces α by its conjugates) are thus linearly independent. But then the vector r is orthogonal to d linearly independent vectors, a contradiction. □

The main issue is now to prove that an irreducible Pisot substitutive dynamical system over d letters does not only admit as a factor a minimal translation on the torus $\mathbb{T}^{d-1}$, but that is measure-theoretically isomorphic with this translation.

Natural codings and bounded remainder sets. Let Λ be a full-rank lattice in $\mathbb{R}^d$ and $T_{\mathbf{t}} : \mathbb{R}^d/\Lambda \to \mathbb{R}^d/\Lambda$, $\mathbf{x} \mapsto \mathbf{x} + \mathbf{t}$ a given toral translation. Let $R \subset \mathbb{R}^d$ be a fundamental domain for Λ and $\tilde{T}_{\mathbf{t}} : R \to R$ the mapping induced by $T_{\mathbf{t}}$ on R. If $R = R_1 \cup \cdots \cup R_k$ is a partition of R (up to measure zero) such that for each $1 \leq i \leq k$ the restriction $\tilde{T}_{\mathbf{t}}|_{R_i}$ is given by the translation $\mathbf{x} \mapsto \mathbf{x} + \mathbf{t}_i$ for some $\mathbf{t}_i \in \mathbb{R}^d$, and u is the coding of a point $\mathbf{x} \in R$ with respect to this partition, we call u a *natural coding* of $T_{\mathbf{t}}$. A symbolic dynamical system (X, Σ) is a *natural coding* of $(\mathbb{R}^d/\Lambda, T_{\mathbf{t}})$ if (X, Σ) and $(\mathbb{R}^d/\Lambda, T_{\mathbf{t}})$ are measure-theoretically isomorphic and every element of X is a natural coding of the orbit of some point of the d-dimensional torus $\mathbb{R}^d/\Lambda$ (with respect to some fixed partition).

Observe that if (X, S) is a natural coding of a minimal translation $(\mathbb{R}^d/\Lambda, T_{\mathbf{t}})$ with balanced language, then the elements of its associated partition are bounded remainder sets [1, Proposition 7]. Moreover, A is a bounded remainder set if it is an atom of a partition that gives rise to a natural coding of a translation whose induced mapping on A is again a translation; see [53] (we also refer to [35] for an analogous characterization of bounded remainder sets).

5 Pisot conjecture and Rauzy fractals

We have seen that Pisot substitutions admit a toral translation factor. It is widely believed that Pisot irreducible substitutions[2] have discrete spectrum: this is called the *Pisot conjecture*. For more details, see e.g. [51], Chap. 7 and [14]. See also in the same vein [57] whose main concern is Pisot automorphisms of the torus (instead of substitutions).

Example 5. Consider as a first example the Fibonacci substitution $\sigma\colon a \mapsto ab,\ b \mapsto a$; (X_σ, S) is measure-theoretically isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{\frac{1+\sqrt{5}}{2}})$.

For more details see e.g. Chap. 5 in [51]. Furthermore, two-letter Pisot substitutions are known to have discrete spectrum [13, 41, 43]. See also [14, 44, 60, 6] for more on Pisot substitutive dynamical systems.

If the Pisot irreducible substitution σ is furthermore assumed to be unimodular, then (X_σ, S) is conjectured to be measure-theoretically isomorphic to a toral translation. One strategy for providing a fundamental domain for the toral translation has been developed by Rauzy in the case of the Tribonacci substitution $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$ in [52]. It is a primitive, unimodular and Pisot irreducible substitution. Its characteristic polynomial is $X^3 - X^2 - X - 1$ and its dominant eigenvalue $\beta > 1$ is a Pisot number.

Theorem 10 ([52])**.** *Let σ be the Tribonacci substitution $\sigma\colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$. Let β be the Perron–Frobenius eigenvalue of σ. Let $R_\beta\colon \mathbb{T}^2 \to \mathbb{T}^2$, $x \mapsto x + (1/\beta, 1/\beta^2)$. The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the toral translation $(\mathbb{T}^2, R_\beta)$.*

The proof makes use of the fact that the Tribonacci sequence $\sigma^\infty(1) = \lim_{n\to\infty} \sigma^n(1)$ codes the orbit of the point 0 under the action of the translation R_β with respect to a particular partition of $\mathbb{T}^2$ (it is a natural coding). In order to get this partition, one constructs a so-called *Rauzy fractal* as follows. One first represents $(u_n)_{n\in\mathbb{N}} = \sigma^\infty(1)$ as a broken line via the abelianization map l

$$l\colon \mathcal{A}^* \to \mathbb{N}^3,\ w \mapsto (|w|_1, |w|_2, |w|_3).$$

The vertices of this broken line belong to $\mathbb{Z}^3$ and are of the form $l(u_0 \cdots u_n)$ for $n \in \mathbb{N}$. We then project the vertices of this broken line according to the eigenspaces of the incidence matrix M_σ, that is, along its expanding line onto its contracting plane. The corresponding projection is denoted by π. The Rauzy fractal associated with σ is then obtained by taking the closure of this set of points, i.e., as

$$\mathcal{R}_\sigma := \overline{\{\pi \circ l(u_0 \cdots u_n)\,;\, n \in \mathbb{N}\}}.$$

[2]Recall that a Pisot substitution is said irreducible if the characteristic polynomial of its incidence matrix is irreducible.

We then divide $\mathcal{R}_\sigma$ into the three pieces defined for $i = 1, 2, 3$ as

$$\mathcal{R}_\sigma(i) := \overline{\{\pi \circ l(u_0 \cdots u_n)\,;\, u_n = i,\ n \in \mathbb{N}\}}.$$

Theorem 10 can be reformulated as follows: the Rauzy fractal $\mathcal{R}_\sigma$ is a fundamental domain of $\mathbb{T}^2$ and $\sigma^\infty(1)$ codes the orbit of the point 0 under the action of the translation R_β with respect to the particular partition $(\mathcal{R}_\sigma(i))_{i=1,2,3}$ of the fundamental domain $\mathcal{R}_\sigma$ of $\mathbb{T}^2$. Figure 1 depicts a Rauzy fractal together with a piece of periodic tiling that illustrates the fact that it is a fundamental domain for the toral translation R_β on $\mathbb{T}^2$.

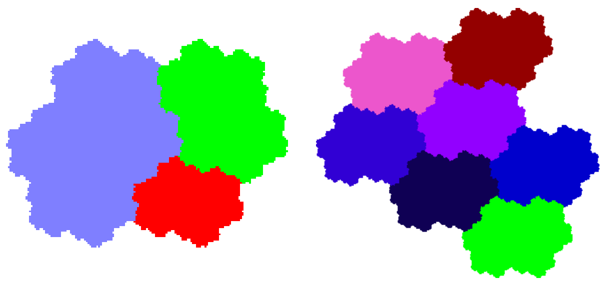

Figure 1: The Rauzy fractal and a piece of the associated periodic tiling.

Rauzy fractals were first introduced in [52] in the case of the Tribonacci substitution, and then in [62], in the case of the β-numeration associated with the Tribonacci number. Rauzy fractals can more generally be associated with Pisot substitutions (see the surveys [51, 16, 6]), as well as with Pisot β-shifts under the name of *central tiles* (see [3, 4, 5]).

A statement generalizing Theorem 10 is conjectured to hold for any Pisot irreducible substitution; note that the corresponding parameters would be algebraic, since they are given by eigenvalues and eigenvectors of the incidence matrix of the substitution. Note also that the subtiles $\mathcal{R}_\sigma(i)$ of the Rauzy fractal are bounded remainder sets for the toral translation R_β (as a consequence of the balance properties).

Theorem 20 will provide an a.e. generalization of Theorem 10. Indeed, thanks to the S-adic formalism that we now introduce, a.e. translation of $\mathbb{T}^2$ admits a symbolic coding.

6 S-adic words

We now extend the Pisot substitutive dynamics to the S-adic framework.

6.1 First definitions

Let $\mathcal{S}$ be a set of substitutions. Let $s = (\sigma_n)_{n\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}}$, with $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$, be a sequence of substitutions, and let $(a_n)_{n\in\mathbb{N}}$ be a sequence of letters with $a_n \in \mathcal{A}_n$ for all n. We say that the infinite word $u \in \mathcal{A}^{\mathbb{N}}$ admits $((\sigma_n, a_n))_n$ as an S*-adic representation* if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1 \cdots \sigma_{n-1}(a_n).$$

The sequence s is called the *directive sequence* and the sequences of letters $(a_n)_n$ will only play a minor role compared to the directive sequence. If the set $\mathcal{S}$ is finite, it makes no difference to consider a constant alphabet (i.e., $\mathcal{A}_n^* = \mathcal{A}^*$ for all n and for all substitution σ in $\mathcal{S}$). As we will constantly use products of substitutions, we introduce the notation $\sigma_{[k,l)}$ for the product $\sigma_k\sigma_{k+1}\ldots\sigma_{k+l-1}$. In particular, $\sigma_{[0,n)} = \sigma_0\sigma_1\ldots\sigma_{n-1}$.

A word admits many possible S-adic representations. But some S-adic representations might be useful to get information about the word. More precisely, some properties are actually equivalent to have some S-adic representation of a special kind (see in particular Theorem 11 for minimality and Theorem 13 for linear recurrence).

In order to avoid degeneracy construction we introduce the following definition. An S-adic representation defined by the directive sequence $(\sigma_n)_{n\in\mathbb{N}}$, where $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$, is *everywhere growing* if for any sequence of letters $(a_n)_n$, one has

$$\lim_{n\to+\infty} |\sigma_{[0,n)}(a_n)| = +\infty.$$

To be "S-adic" is not an intrinsic property of an infinite word, but a way to construct it. Indeed, without further restriction, every infinite word $u = (u_n)_n \in \mathcal{A}^{\mathbb{N}}$ admits an S-adic representation. Here, we recall the classical construction due to J. Cassaigne. We consider $u = (u_n)_n \in \mathcal{A}^{\mathbb{N}}$. We introduce a further letter $\ell \notin \mathcal{A}$, and we work on $\mathcal{A} \cup \{\ell\}$. For every letter $a \in \mathcal{A}$, we introduce the substitution σ_a defined over the alphabet $\mathcal{A} \cup \{\ell\}$ as $\sigma_a(b) = b$, for $b \in \mathcal{A}$, and $\sigma_a(\ell) = \ell a$. We also consider the substitution τ_ℓ over the alphabet $\mathcal{A} \cup \{\ell\}$ defined as $\tau_\ell(\ell) = u_0$, and $\tau_\ell(b) = b$, for all $b \in \mathcal{A}$. One checks that

$$u = \lim_{n\to+\infty} \tau_\ell\, \sigma_{u_1}\, \sigma_{u_1}\, \ldots\, \sigma_{u_n}(\ell).$$

Hence u admits an S-adic representation with $\mathcal{S} = \{\sigma_a \mid a \in \mathcal{A}\} \cup \{\tau_\ell\}$. We stress the fact that, despite u belongs to $\mathcal{A}^{\mathbb{N}}$, this S-representation involves the larger-size alphabet $\mathcal{A} \cup \{\ell\}$. Observe also that this representation is not everywhere growing.

Given an everywhere growing directive sequence s of substitutions that are all defined over the same finite alphabet $\mathcal{A}$, the *shift* associated with $s = (\sigma_n)_n$ is the set of infinite words whose language (i.e., whose set of factors) is included in the intersection of languages

$$\bigcap_n \mathcal{L}_{\sigma_0\cdots\sigma_{n-1}(\mathcal{A})}.$$

6.2 Sturmian words

We recall the definition of Sturmian words for which the Fibonacci word is a particular case. We consider the substitutions τ_a and τ_b defined over the alphabet $\mathcal{A} = \{a, b\}$ by $\tau_a\colon a \mapsto a, b \mapsto ab$ and $\tau_b\colon a \mapsto ba, b \mapsto b$. Let $(i_n) \in \{a, b\}^{\mathbb{N}}$. The following limits

$$u = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1}\cdots\tau_{i_{n-1}}(a) = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1}\cdots\tau_{i_{n-1}}(b) \qquad (1)$$

exist and coincide whenever the directive sequence $(i_n)_n$ is not ultimately constant (it is easily shown that the shortest of the two images by $\tau_{i_0}\tau_{i_1}\ldots\tau_{i_{n-1}}$ is a prefix of the other).

The infinite words thus produced belong to the class of Sturmian words. More generally, a *Sturmian word* is an infinite word whose set of factors coincides with the set of factors of a sequence u of the form (1), with the sequence $(i_n)_{n\geq 0}$ being not ultimately constant (that is, it is an elementt of the symbolic dynamical system X_u generated by u, since (X_u, S) is minimal).

Let us consider a second set of substitutions $\mu_a\colon a \mapsto a, b \mapsto ba$ and $\mu_1\colon a \mapsto ab, b \mapsto b$. In this latter case, the two corresponding limits do exist, namely

$$w_a = \lim_{n\to\infty} \mu_{i_0}\mu_{i_1}\cdots\mu_{i_n}(a) \quad \text{and} \quad w_b = \lim_{n\to\infty} \mu_{i_0}\mu_{i_1}\cdots\mu_{i_n}(b)$$

for any sequence $(i_n) \in \{a, b\}^{\mathbb{N}}$. As w_a starts with a and w_b with b they do not coincide. But, provided the directive sequence $(i_n)_n$ is not ultimately constant, the languages generated by w_a and w_b are the same, and they also coincide with the language generated by u (for the same sequence (i_n)).

6.3 Arnoux-Rauzy words

Arnoux and Rauzy [7] proposed a generalization of Sturmian words to higher size alphabets (which initiated an important literature around so-called episturmian words).

Let $\mathcal{A} = \{1, 2, \ldots, d\}$. The set of Arnoux-Rauzy substitutions is defined as $\mathcal{S}_{AR} = \{\mu_i \mid i \in \mathcal{A}\}$ where

$$\mu_i :\ i \mapsto i,\ j \mapsto ji \text{ for } j \in \mathcal{A} \setminus \{i\}\,.$$

One recovers Sturmian words in the case $d = 2$. An *Arnoux-Rauzy word* [7] is an infinite word $\omega \in \mathcal{A}^{\mathbb{N}}$ whose set of factors coincides with the set of factors of a sequence of the form

$$\lim_{n\to\infty} \mu_{i_0}\mu_{i_1}\cdots\mu_{i_n}(1),$$

where the sequence $(i_n)_{n\geq 0} \in \mathcal{A}^{\mathbb{N}}$ is such that every letter in $\mathcal{A}$ occurs infinitely often in $(i_n)_{n\geq 0}$. For more on Arnoux-Rauzy words, see [29, 28].

It was conjectured since the early nineties (see e.g. [28]) that each Arnoux-Rauzy word is a natural coding of a translation on the torus. A counterexample to this conjecture was provided in [28] by constructing unbalanced Arnoux-Rauzy words (unbalanced words cannot come from natural codings by a result of

Rauzy [53]). Moreover, [29] even showed that there exist Arnoux-Rauzy words u on three letters such that (X_u, S) is weakly mixing (w.r.t. the unique S-invariant probability measure on X_u). Theorem 17 below confirms the conjecture of Arnoux and Rauzy almost everywhere.

6.4 Minimality and linear recurrence

In this section we introduce two notions of primitivity for S-adic expansions. We relate them respectively to minimality (Theorem 11) and linear recurrence (Theorem 12 and 13).

Definition 1 (Primitivity)**.** An S-adic expansion with directive sequence $(\sigma_n)_n$ is said *weakly primitive* if, for each n, there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive.

It is said *strongly primitive* if the set of substitutions $\{\sigma_n\}$ is finite, and if there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive, for each n.

Theorem 11. *If an infinite word u admits a weakly primitive S-adic representation, then it is uniformly recurrent (and the shift X_u is minimal).*

Proof. Let $(\sigma_n)_n$ be the weakly primitive directive sequence of substitutions of an S-adic representation of u. Observe that this representation is necessarily everywhere growing. Consider a factor w of the language. It occurs in some $\sigma_{[0,n)}(i)$ for some integer $n \geq 0$ and some letter $i \in \mathcal{A}$. By definition of weak primitivity, there exists an integer r such that $\sigma_{[n,n+r)}$ is positive. Hence w appears in all images of letters by $\sigma_{[0,n+r)}$ which implies uniform recurrence. $\square$

The following statement from [33] illustrates the fact that strong primitivity plays the role of primitivity in the S-adic context.

Theorem 12 ([33])**.** *Let u be an S-adic word having a strongly primitive S-adic expansion. Then, the associated shift (X_u, T) is minimal (that is, u is uniformly recurrent), uniquely ergodic, and it has at most linear factor complexity.*

Strong primitivity is thus closely related to linear recurrence, that can be considered as a property lying in between being substitutive and being S-adic. With an extra condition of properness one even obtains the following characterization of linear recurrence. A substitution over $\mathcal{A}$ is said *proper* if there exist two letters $b, e \in \mathcal{A}$ such that for all $a \in \mathcal{A}$, $\sigma(a)$ begins with b and ends with e. An S-adic system is said to be *proper* if the substitutions in $\mathcal{S}$ are proper.

Theorem 13 (Linear recurrence [33])**.** *A subshift (X, T) is linearly recurrent if and only if it is a strongly primitive and proper S-adic subshift.*

Let us mention that an essential ingredient in the proofs of Theorem 12 and Theorem 13 is the uniform growth of the matrices $M_{(0,n)}$ as it was the case for substitutive systems. We refer to [33] for the proof.

With the following example, we stress the fact that strong primitivity alone does not imply linear recurrence. Indeed, linear recurrence requires the property of being also proper S-adic.

Example 6. We recall the example of [33] of a strongly primitive S-adic word, that is both uniformly recurrent, that has linear factor complexity, but that is not linearly recurrent. Take $S = \{\sigma, \tau\}$ with $\sigma\colon a \mapsto acb,\ b \mapsto bab,\ c \mapsto cbc$, $\tau\colon a \mapsto abc,\ b \mapsto acb,\ c \mapsto aac$, and consider the S-adic expansion

$$\lim_{n\to+\infty} \sigma\,\tau\,\sigma\,\tau\,\cdots\,\sigma^n\tau(a).$$

6.5 Invariant measures

This section is devoted to frequencies of S-adic systems, and more generally, to their invariant measures. We use the following notation: M_n stands for the incidence matrix of σ_n, and $M_{[0,n)} = M_0 M_1 \cdots M_{n-1}$.

S-adic representations prove to be convenient to find invariant measures. Indeed, given a directive sequence $(\sigma_n)_n$ that is everywhere growing, the *limit cone* determined by the incidence matrices of the substitutions σ_n, namely

$$\bigcap_n M_{[0,n)}\mathbb{R}^d_+,$$

is intimately related to letter frequencies in the corresponding S-adic shift: it is the convex hull of the set of half lines $\mathbb{R}_+ f$ generated by the letter frequency vectors f of words in the associated S-adic shift. Nevertheless, the situation is more contrasted for S-adic systems than for substitutive dynamical systems, for which primitivity implies unique ergodicity (see Theorem 1). This is well known since Keane's counterexample for unique ergodicity for 4-interval exchanges [46]: weak primitivity does not imply unique ergodicity.

Recall that for a primitive matrix M, the cones $M^n\mathbb{R}^d_+$ nest down to a single line directed by this eigenvector at an exponential convergence speed, according to the Perron-Frobenius Theorem (see e.g. [59]). The following condition is a sufficient condition for the sequence of cones $M_{[0,n)}\mathbb{R}^d_+$ to nest down to a single strictly positive direction as n tends to infinity (provided that the square matrices M_n have all non-negative entries); in other words, the columns of the product $M_{[0,n)}$ tend to be proportional.

Theorem 14 ([37, pp. 91–95])**.** *Let $(M_n)_n$ be a sequence of non-negative integer matrices of size d and note $M_{[0,n)} = M_0 M_1 \ldots M_{n-1}$. Assume that there exist a strictly positive matrix B and indices $j_1 < k_1 \le j_2 < k_2 \le \cdots$ such that $B = M_{j_1} \cdots M_{k_1-1} = M_{j_2} \cdots M_{k_2-1} = \cdots$. Then,*

$$\bigcap_{n\in\mathbb{N}} M_{[0,n)}\mathbb{R}^d_+ = \mathbb{R}_+ f \quad \textit{for some positive vector } f \in \mathbb{R}^d_+.$$

The proof of that theorem relies on classical methods for non-negative matrices, namely Birkhoff contraction coefficient estimates and projective Hilbert metric [22].

This vector f, when normalized so that the sum of its coordinates equals 1, is called the *generalized right eigenvector* associated with the S-adic representation. Note that there is no way to define a left eigenvector as the sequence of rows vary dramatically in the sequence of matrices $(M_{[0,n)})_n$.

Now, we consider frequencies of words or, equivalently, invariant measures.

Theorem 15. *Let X be an S-adic shift with directive sequence $\tau = (\tau_n)_n$ where $\tau_n \colon \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ and $\mathcal{A}_0 = \{1, \ldots, d\}$. Denote by $(M_n)_n$ the associated sequence of incidence matrices.*

If the sequence $(\tau_n)_n$ is everywhere growing, then the cone

$$C^{(0)} = \bigcap_{n \to \infty} M_{[0,n)} \mathbb{R}_+^d$$

parametrizes the letter frequencies: the set of vectors $f \in C^{(0)}$ such that $f_1 + \ldots + f_d = 1$ coincides with the image of the map which sends a shift-invariant probability measure μ on X to the vector of letter frequencies $(\mu([1]), \mu([2]), \ldots, \mu([d]))$. In particular X has uniform letter frequencies if and only if the cone $C^{(0)}$ is one-dimensional.

If, furthermore, the limit cone

$$C^{(k)} = \bigcap_{n \to \infty} M_{[k,n)} \mathbb{R}_+^d$$

is one-dimensional, then the S-adic dynamical system (X, T) is uniquely ergodic.

Note that if the matrices are invertible, then $C^{(0)}$ is one-dimensional if and only if $C^{(k)}$ is one-dimensional for any k. In [21], a somewhat finer version of Theorem 15 is proved in the context of Bratteli diagrams where a similar limit cone is identified to the set of invariant ergodic probability measures. For more on the connections between Vershik adic systems and S-adic ones, see [16, Chap. 6].

7 Pisot S-adic shifts

We now introduce an S-adic counterpart to the notion of Pisot substitution.

7.1 Lyapunov exponents and convergence

Let $\mathcal{S}$ be a finite set of substitutions with invertible incidence matrices, and let (D, S, μ) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an (ergodic) shift equipped with a probability measure μ. Here again S stands for the shift acting on D. Given an infinite sequence of substitutions $\gamma = (\gamma_n)_n \in D$, we define

$$A_n(\gamma) = M_{\gamma_0} M_{\gamma_1} \ldots M_{\gamma_{n-1}},$$

where M_i is the incidence matrix of the substitution γ_i. In particular, $A_1(\gamma) = M_{\gamma_0}$ and we have the following *cocycle relation* (recall that S stands for the shift):

$$A_{m+n}(\gamma) = A_m(\gamma) A_n(S^m \gamma).$$

The Lyapunov exponents of the cocycle A_n with respect to the ergodic probability measure μ provide the exponential growth of eigenvalues of the matrices

A_n along a μ-generic sequence γ. Lyapunov exponents were first defined by Furstenberg [38, 37], and, in a sense, their existence generalizes the Birkhoff ergodic Theorem in a non-commutative setting. For general references on Lyapunov exponents, we refer to [23] and [36]. We recall that the incidence matrices of the substitutions in $\mathcal{S}$ are assumed to be invertible (in other words $A_1(\gamma)$ belongs to $\mathrm{GL}(d,\mathbb{R})$). We say that the cocycle A_n is log-*integrable* if

$$\int_{\Sigma_G} \log\max(\|A_1(\gamma)\|, \|A_1(\gamma)^{-1}\|)d\mu(\gamma) < \infty.$$

Since the matrices $A_1(\gamma)$ are bounded (the set $\mathcal{S}$ is finite), this condition is automatically satisfied. When the matrices are not invertible, or without log-integrability, one may obtain infinite Lyapunov exponents.

Assuming the ergodicity of μ and the log-integrability of A_1, the first Lyapunov exponent of (D, S, ν) is the μ-a.e. limit

$$\theta_1^\mu = \lim_{n\to\infty} \frac{\log\|A_n(\gamma)\|}{n}.$$

The other Lyapunov exponents $\theta_2^\mu \geq \theta_3^\mu \ldots \geq \theta_d^\mu$ may be defined recursively by the following almost everywhere limits, for $k = 1, \ldots, d$:

$$\theta_1^\mu + \theta_2^\mu + \cdots + \theta_k^\mu = \lim_{n\to\infty} \frac{\log\| \wedge^k A_n(\gamma)\|}{n}$$

where $\wedge^k$ stands for the k-th exterior product.

We will mostly be interested by the two first Lyapunov exponents θ_1^μ and θ_2^μ. A useful characterization of θ_2^μ is as follows. Assume that for a.e. γ the sequence of nested cones $(A_n(\gamma)\mathbb{R}_+^d)_n$ converges to a line $f(\gamma)$. Then, we have the μ-almost everywhere limit

$$\theta_2^\mu = \lim_{n\to\infty} \frac{\log\|A_n|_{f(\gamma)^\perp}\|}{n} \tag{2}$$

where $\|A_n(\gamma)|_{f(\gamma)^\perp}\| = \sup_{v\in f(\gamma)^\perp} \frac{\|A_n v\|}{\|v\|}$.

7.2 Simultaneous approximation and cone convergence

If we follow the vocabulary of Markov chains [59], or of continued fractions [24, 58], it is natural to consider the following definitions.

Definition 2 (Weak and strong convergence)**.** Let X be an S-adic shift with directive sequence $\tau = (\tau_n)_n$. Denote by $(M_n)_n$ the associated sequence of incidence matrices. We assume that the hypotheses of Theorem 14 hold. Let f be the generalized right eigenvector on the alphabet $\mathcal{A} = \{1, \ldots, d\}$ (normalized by $f_1 + \ldots + f_d = 1$). Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$. The S-adic system X is *weakly convergent* toward the non-negative half-line directed by f if

$$\forall i \in \{1, \ldots, d\}, \qquad \lim_{n\to\infty} \mathrm{d}\left(\frac{M_0 \cdots M_{n-1} e_i}{\|M_0 \cdots M_{n-1} e_i\|_1}, f\right) = 0.$$

It is said to be *strongly convergent* if for a.e. f

$$\forall i \in \{1, \ldots, d\}, \qquad \lim_{n\to\infty} \mathrm{d}(M_0 \cdots M_{n-1} e_i, \mathbb{R} f) = 0.$$

If, for any i, $\|M_0 \cdots M_{n-1} e_i\|$ tends to infinity as n tends to infinity, then the strong convergence for the vector f is equivalent to the fact that the sequence of nested cones $(M_0 \cdots M_{n-1} \mathbb{R}_+^d)_n$ tends to the non-negative half-line generated by f: the points $M_0 \cdots M_{n-1} e_i / \|M_0 \cdots M_{n-1} e_i\|_1$ are exactly the extremal points of the intersection of the cone $M_0 \cdots M_{n-1} \mathbb{R}_+^d$ with the set of vectors of norm 1. Moreover, if $\theta_{i,n}$ is the angle between $M_0 \cdots M_{n-1} e_i$ and f, then

$$\mathrm{d}\left(\frac{M_0 \cdots M_{n-1} e_i}{\|M_0 \cdots M_{n-1} e_i\|_1}, f\right) = 2 \sin\left(\frac{\theta_{i,n}}{2}\right)$$

and

$$\frac{\mathrm{d}(M_0 \cdots M_{n-1} e_i, \mathbb{R} f)}{\|M_0 \cdots M_{n-1} e_i\|_2} = \sin(\theta_{i,n}).$$

A finite product of substitutions $\gamma_0 \ldots \gamma_{k-1}$ is said *positive* if the associated matrix $M_{\gamma_0} \ldots M_{\gamma_{k-1}}$ is positive. Recall that $A_n(\gamma) = M_{\gamma_0} \cdots M_{\gamma_{n-1}}$.

Theorem 16. *Let $\mathcal{S}$ be a finite set of substitutions with invertible incidence matrices, and let (D, S, μ), with $D \subset \mathcal{S}^{\mathbb{N}}$, be an argodic shift. Assume that there exists a positive product of substitutions whose associated cylinder has positive mass for μ. Then, for μ-almost every sequence $\gamma \in D$, the corresponding S-adic system $X_D(\gamma)$ is uniquely ergodic. Furthermore, one has*

$$\theta_1^\mu > 0 \text{ and } \theta_1^\mu > \theta_2^\mu.$$

Let $f(\gamma) = (f_i(\gamma))_{i \in \mathcal{A}}$ denote the generalized right eigenvector of a μ-generic sequence γ. For μ-almost every S-adic sequence in D, $X_D(\gamma)$ is weakly convergent:

$$\forall i \in \{1, \ldots, d\}, \qquad \lim_{n\to\infty} \frac{1}{n} \log \mathrm{d}\left(\frac{A_n(\gamma) e_i}{\|A_n(\gamma) e_i\|_1}, f(\gamma)\right) = \theta_2^\mu - \theta_1^\mu.$$

Moreover, if $\theta_2^\mu < 0$, then, for μ-almost every S-adic sequence in D, $X_D(\gamma)$ is strongly convergent:

$$\lim_{n\to\infty} \max_{i \in \mathcal{A}} \frac{\log \mathrm{d}(A_n(\gamma) e_i, \mathbb{R} f(\gamma))}{n} = \theta_2^\mu,$$

and for μ-almost all γ in D, there exists a constant $C = C(\gamma)$ such that for every letter $i \in \mathcal{A}$, every word u in $X_D(\gamma)$ and every n, we have

$$||u_0 \ldots u_{n-1}|_i - n f_i| \leq C.$$

In particular, each word in $X_D(\gamma)$ is C-balanced.

Proof. We only sketch the proof. According to Theorem 15, uniform letter frequencies for the S-adic system associated with $\gamma \in D$ is equivalent to the fact that the sequence of nested cones $(A_n(\gamma)\mathbb{R}^d_+)_n$ tends to a half-line $\mathbb{R}f(\gamma)$. Let $B = M_{\gamma_0} \dots M_{\gamma_{k-1}}$ be the positive matrix associated with the positive product $\gamma_0 \dots \gamma_{k-1}$ which has positive mass. We can use the Birkhoff ergodic Theorem to see that μ-almost every sequence D contains infinitely often the factor $\gamma_0 \dots \gamma_{k-1}$. We can hence apply Theorem 14, and get that $A_n(\gamma)\mathbb{R}^d_+$ contracts almost everywhere to a cone. The existence of the positive path with positive mass thus implies that μ-almost every S-adic path in D gives a uniquely ergodic S-adic system. Moreover, because of the non-negativity of the matrices, we get that the exponential growth of $\log \|A_n(\gamma)\|$ is at least of the order of $\log \|B^{\lfloor n\mu([\gamma_0 \cdots \gamma_{k-1}])\rfloor}\|$. It follows that $\theta_1^\mu > 0$.

To prove that $\theta_1^\mu > \theta_2^\mu$, we only need to remark that B induces a contraction of the Hilbert metric (by positivity). Therefore, the sequence of nested cones $A_n(\gamma)\mathbb{R}^d_+$ shrinks exponentially fast toward the generalized eigendirection. As it can be seen with (2), this exponential rate is $\theta_2^\mu - \theta_1^\mu < 0$.

For the proof of the convergence properties, see [47] where a similar result is shown in the context of Diophantine approximation.

We only sketch the proof of the balance properties which is very similar to what is done in [65] or [2]. We have seen that letters have uniform frequencies. Let $\gamma = (\gamma_n)_n$ be a generic directive sequence of substitutions, X_γ the associated shift space, and f the associated letter frequency vector. Let $W_n = \{\gamma_{[0,n)}(a) \mid a \in \mathcal{A}\}$ be the set of images of letters by $\gamma_{[0,n)} = \gamma_0 \dots \gamma_{n-1}$. Then, by definition of Lyapunov exponents, for every $\varepsilon > 0$, for n large enough, for all $w \in W_n$ and all $i \in \mathcal{A}$, we have

$$\exp(n(\theta_1^\mu - \varepsilon) \leq |w| \leq \exp(n(\theta_1^\mu + \varepsilon))$$

and

$$||w|_i - |w|f_i| \leq \exp(n(\theta_2^\mu + \varepsilon)).$$

In particular, we get that for $w \in W_n$ with n large enough

$$\frac{\log(|w|_i - |w|f_i|)}{\log(|w|)} \leq \frac{\theta_2^\mu + \varepsilon}{\theta_1^\mu - \varepsilon}.$$

This proves that the balancedness property holds for the elements of W_n. Now any word in X may be decomposed with respect to the building blocks W_n (according to the Dumont-Thomas prefix-suffix decomposition [32]). From that decomposition, it remains to perform a summation to obtain the theorem. $\square$

Note that the quantity $1 - \frac{\theta_2^\mu}{\theta_1^\mu} = \frac{1}{\theta_1^\mu}(\theta_1^\mu - \theta_2^\mu)$ is expressed in [47] as the uniform approximation exponent for unimodular continued fractions algorithms such as the Jacobi-Perron algorithm (the coefficient $1/\theta_1^\mu$ is here to take care of the size of the denominators); see also [10, 11] in the same vein.

As already discussed in Section 3.2, a characterization of balanced words generated by primitive substitutions is given in [1, Corollary 15]. It is shown that there exist balanced fixed points of substitutions for which $\theta_2 = 0$, i.e., the incidence matrix of the substitution has an eigenvalue of modulus one. It is even proved that if a primitive substitution generates an infinite word that is balanced, then all eigenvalues of modulus one of the incidence matrix have to be roots of unity.

7.3 S-adic Pisot shifts

We now can introduce the S-adic counterpart of the notion of irreducible substitution. Recall that a substitution is said *irreducible Pisot* if the characteristic polynomial of its incidence matrix is the minimal polynomial of a Pisot number, that is, a real algebraic integer larger than 1 whose other Galois conjugates are smaller than 1 in modulus.

Let $\mathcal{S}$ be a finite set of substitutions with invertible incidence matrices, and let (D, S, μ) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an (ergodic) shift equipped with a probability measure μ. We say that (D, S, μ) satisfies the *Pisot condition* if

$$\theta_1^\mu > 0 > \theta_2^\mu.$$

By Theorem 16, an S-adic shift (satisfying the general assumptions of the theorem) endowed with a measure μ such that the Lyapunov exponents satisfy the Pisot condition is such that for μ-almost every directive sequence γ, the associated S-adic system $X_D(\gamma)$ is made of balanced words. This property is known to hold for some continued fraction algorithms endowed with their absolute continuous measure: the standard continued fractions, the Brun algorithm in dimension 3 [27], the Jacobi-Perron algorithm in dimension 3 [25, 26]. Some more precise results that hold for *all* measures are proven in [8].

The analogs of primitivity and algebraic irreducibility are then the following. The directive sequence γ is said to be *algebraically irreducible* if, for each $k \in \mathbb{N}$, the characteristic polynomial of $M_{[k,\ell)}$ is irreducible for all sufficiently large ℓ. Recall that the the sequence γ is said to be (weakly) *primitive* if, for each $k \in \mathbb{N}$, $M_{[k,\ell)}$ is a positive matrix for *some* $\ell > k$.

7.4 Rauzy fractals

We now have gathered all the required elements in order to define a Rauzy fractal in the framework of Pisot S-adic shifts.

Assume we are given a (weakly) primitive directive sequence γ. Let (X_γ, S) be the shift generated by γ and $\mathcal{L}_\gamma$ stand for its language. Let u be an infinite word in X_γ of the form

$$u = \lim \gamma_0 \cdots \gamma_n(i_n).$$

Such a word is called a *limit word* for the directive sequence γ and exists by primitivity. The set of factors of u coincides with the language $\mathcal{L}_\gamma$. Assume that the conditions of Theorem 14 hold. Let f be the generalized right eignevector of γ.

The Rauzy fractal $\mathcal{R}$ is defined as the closure of the projection of the vertices of the broken lines defined by limit words of γ. More precisely, let $1^{\perp}$ be the hyperplane orthogonal to the vector with entries all equal to 1, that is, $1^{\perp}$ is the hyperplane of vectors whose entries sum up to 0.

The *Rauzy fractal* (in the representation space $1^{\perp}$) associated with the directive sequence of substitutions γ over the alphabet $\mathcal{A}$ with generalized right eigenvector f is

$$\mathcal{R}_\gamma = \overline{\{\pi \circ l(u_0 \cdots u_n)\,;\, n \in \mathbb{N}\}},$$

where π denotes the projection along the direction of f onto $1^{\perp}$. The Rauzy fractal has natural *subpieces* (or *subtiles*) defined by

$$\mathcal{R}_\gamma(i) = \overline{\{\pi \circ l(u_0 \cdots u_n)\,;\, u_n = i,\ n \in \mathbb{N}\}}.$$

Note that we choose to project here onto the plane $1^{\perp}$, whereas in Section 5 we project onto the contracting plane of M_σ. There is no well-defined notion of contracting plane in the present context, hence our projection choice. Note also that under the assumptions we will use below, $\mathcal{R}_\gamma$ will not depend on the choice of the limit word u.

The next statement shows that the Rauzy fractal $\mathcal{R}$ corresponding to a sequence γ is bounded if $\mathcal{L}_\gamma$ is balanced. Therefore, $\mathcal{R}$ is compact if the broken lines provided by limit words remain at a bounded distance from the generalized right eigendirection $\mathbb{R}f$. It establishes a connection between the degree of balancedness and the diameter of $\mathcal{R}$. Here $\|\cdot\|$ denotes the maximum norm. For the proof, see [19].

Lemma 3 ([19]). *Let γ be a (weakly) primitive sequence of substitutions with generalized right eigenvector f. Then $\mathcal{R}$ is bounded if and only if $\mathcal{L}_\gamma$ is balanced. If $\mathcal{L}_\gamma$ is C-balanced, then $\mathcal{R} \subset [-C, C]^d \cap 1^{\perp}$.*

Irrationality and strong convergence In the periodic case with a unimodular irreducible Pisot substitution σ, the incidence matrix M_σ has an expanding right eigenline and a contractive right hyperplane (that is orthogonal to an expanding left eigenvector), i.e., the matrix M_σ contracts the space $\mathbb{R}^d$ towards the expanding eigenline. Moreover, irreducibility implies that the coordinates of the expanding eigenvector are rationally independent. These properties are crucial for proving that the Rauzy fractal $\mathcal{R}$ has positive measure and induces a (multiple) tiling of the representation space $1^{\perp}$. In the S-adic setting, the cones $M_{[0,n)}\,\mathbb{R}^d_+$ converge weakly to the direction of the generalized right eigenvector f. For the proof, see again [19].

Lemma 4 ([19]). *Let γ be an algebraically irreducible sequence of substitutions with generalized right eigenvector f and balanced language $\mathcal{L}_\gamma$. Then the coordinates of f are rationally independent.*

Now, in order to set up a representation map from X_γ to $\mathcal{R}$, we also define refinements of the subtiles of $\mathcal{R}$ by

$$\mathcal{R}(w) = \overline{\{\pi(l(p)) :\, p \in \mathcal{A}^*,\ pw \text{ is a prefix of a limit word of } \gamma\}} \quad (w \in \mathcal{A}^*).$$

Let γ be a primitive, algebraically irreducible, and recurrent sequence of substitutions with balanced language $\mathcal{L}_\gamma$. Then one checks (see [19]) that $\bigcap_{n\in\mathbb{N}} \mathcal{R}(\zeta_0\zeta_1\cdots\zeta_{n-1})$ is a single point in $\mathcal{R}$ for each infinite word $\zeta_0\zeta_1\cdots \in X_\gamma$. Therefore, the representation map

$$\varphi:\, X_\gamma \to \mathcal{R},\ \zeta_0\zeta_1\cdots \mapsto \bigcap_{n\in\mathbb{N}} \mathcal{R}(\zeta_0\zeta_1\cdots\zeta_{n-1}),$$

is well-defined, continuous and surjective.

As applications of this construction, we will see in the next section, with the examples of Arnoux-Rauzy and Brun S-adis systems, how to deduce a.e. spectral properties.

8 S-adic shifts associated with continued fraction algorithms

8.1 Arnoux-Rauzy words

Using Rauzy fractals, it is possible to obtain the following result that extends-Theorem 10 to a non-algebraic setting.

Theorem 17 ([19]). *Let $\mathcal{S}_{AR}$ be the set of Arnoux-Rauzy substitutions over three letters and consider the shift $(\mathcal{S}_{AR}^{\mathbb{N}}, S, \nu)$ for some shift invariant ergodic probability measure ν which assigns positive measure to each cylinder. Then $(\mathcal{S}_{AR}^{\mathbb{N}}, S, \nu)$ satisfies the Pisot condition. Moreover, for ν-almost all sequences $\gamma \in S^{\mathbb{N}}$ the S-adic shift (X_γ, S) is measure-theoretically isomorphic to a translation on the torus $\mathbb{T}^2$.*

As an example of measure satisfying the assumptions of Theorem 17, consider the measure of maximal entropy for the suspension flow of the Rauzy gasket constructed in [9].

One can also provide a (uncountable) class of non-substitutive Arnoux-Rauzy words that give rise to translations on the torus $\mathbb{T}^2$. To this end we introduce a terminology that comes from the associated Arnoux-Rauzy continued fraction algorithm (which was also defined in [7]). A directive sequence $\gamma = (\gamma_n) \in S^{\mathbb{N}}$ that contains each α_i $(i = 1, 2, 3)$ infinitely often is said to have *bounded weak partial quotients* if there is $h \in \mathbb{N}$ such that $\gamma_n = \gamma_{n+1} = \cdots = \gamma_{n+h}$ does not hold for any $n \in \mathbb{N}$, and *bounded strong partial quotients* if every substitution in the directive sequence γ occurs with bounded gap.

Theorem 18. *Let $\mathcal{S}_{AR} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the set of Arnoux-Rauzy substitutions over three letters. If $\gamma \in \mathcal{S}^{\mathbb{N}}$ is recurrent, contains each α_i $(i = 1, 2, 3)$ infinitely often and has bounded weak partial quotients, then the S-adic shift (X_γ, S) is measure-theoretically isomorphic to a translation on the torus $\mathbb{T}^2$.*

Boundedness of the strong partial quotients provides a nice characterization of linear recurrence for Arnoux-Rauzy words (see Proposition 6 below). With the extra assumption of recurrence (not only on letters but on any factor) of the directive sequence, we obtain pure discrete spectrum.

Corollary 1. Any linearly recurrent Arnoux-Rauzy word with recurrent directive sequence generates a symbolic dynamical system that has pure discrete spectrum.

These last two statements are consequences of the following results.

Proposition 5 ([17, Theorem 7 and its proof])**.** *Let $\gamma \in \{\alpha_1, \alpha_2, \alpha_3\}^{\mathbb{N}}$. If each α_i occurs infinitely often in γ and if we do not have $\gamma_n = \gamma_{n+1} = \cdots = \gamma_{n+h}$ for any $n \in \mathbb{N}$, then $\mathcal{L}_\gamma^{(n)}$ is $(2h+1)$-balanced for each $n \in \mathbb{N}$.*

Proposition 6. *An Arnoux-Rauzy word is linearly recurrent if and only if it has bounded strong partial quotients, that is, each substitution of $\mathcal{S}_{AR}$ occurs in its directive sequence with bounded gaps.*

Proof. Let u be an Arnoux-Rauzy word with directive sequence γ. It is easy to check that strong partial quotients have to be bounded for an Arnoux-Rauzy word u to be linearly recurrent. The converse is a direct consequence of [33] by noticing that the largest difference between two consecutive occurrences of a word of length 2 in $u^{(n)}$ is bounded (with respect to n), where $u^{(n)}$ is associated with $S^n(\gamma)$. □

8.2 Brun words

Let $\Delta_2 := \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le x_2 \le 1\}$ be equipped with the Lebesgue measure λ_2. Brun [27] devised a generalized continued fraction algorithm for vectors $(x_1, x_2) \in \Delta_2$. This algorithm (in its additive form) is defined by the mapping $T_{\mathrm{Brun}} : \Delta_2 \to \Delta_2$,

$$T_{\mathrm{Brun}} : (x_1, x_2) \mapsto \begin{cases} \left(\frac{x_1}{1-x_2}, \frac{x_2}{1-x_2}\right), & \text{for } x_2 \le \frac{1}{2}, \\ \left(\frac{x_1}{x_2}, \frac{1-x_2}{x_2}\right), & \text{for } \frac{1}{2} \le x_2 \le 1 - x_1, \\ \left(\frac{1-x_2}{x_2}, \frac{x_1}{x_2}\right), & \text{for } 1 - x_1 \le x_2; \end{cases} \tag{3}$$

for later use, we define $B(i)$ to be the set of $(x_1, x_2) \in \Delta_2$ meeting the restriction in the i-th line of (3), for $1 \le i \le 3$. An easy computation shows that the linear (or "projectivized") version of this algorithm is defined for vectors $w^{(0)} = (w_1^{(0)}, w_2^{(0)}, w_3^{(0)})$ with $0 \le w_1^{(0)} \le w_2^{(0)} \le w_3^{(0)}$ by the recurrence $M_{i_n} w^{(n)} = w^{(n-1)}$, where M_{i_n} is chosen among the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \tag{4}$$

according to the magnitude of $w_3^{(n-1)} - w_2^{(n-1)}$ compared to $w_1^{(n-1)}$ and $w_2^{(n-1)}$. More precisely, we have

$$T_{\mathrm{Brun}}\big(w_1^{(n-1)}/w_3^{(n-1)}, w_2^{(n-1)}/w_3^{(n-1)}\big) = \big(w_1^{(n)}/w_3^{(n)}, w_2^{(n)}/w_3^{(n)}\big).$$

We associate S-adic words with this algorithm by defining the *Brun substitutions*

$$\beta_1 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 23 \\ 3 \mapsto 3 \end{cases} \qquad \beta_2 : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \qquad \beta_3 : \begin{cases} 1 \mapsto 3 \\ 2 \mapsto 1 \\ 3 \mapsto 23 \end{cases} \tag{5}$$

whose incidence matrices coincide with the three matrices in (4) associated with Brun's algorithm. We prove the following result on the related S-adic words.

Theorem 19. *Let $\mathcal{S}_{BR} = \{\beta_1, \beta_2, \beta_3\}$ be the set of Brun substitutions over three letters, and consider the shift $(\mathcal{S}^{\mathbb{N}}, S, \nu)$ for some shift invariant ergodic probability measure ν that assigns positive measure to each cylinder. Then $(\mathcal{S}_{BR}^{\mathbb{N}}, S, \nu)$ satisfies the Pisot condition. Moreover, for ν-almost all sequences $\gamma \in \mathcal{S}_{BR}^{\mathbb{N}}$, the S-adic shift (X_γ, S) is measure-theoretically isomorphic to a translation on the torus $\mathbb{T}^2$.*

This result implies that the S-adic shifts associated with Brun's algorithm provide a natural coding of almost all rotations on the torus $\mathbb{T}^2$. Indeed, by the (weak) convergence of Brun's algorithm for almost all $(x_1, x_2) \in \Delta_2$ (w.r.t. to the two-dimensional Lebesgue measure; see e.g. [27]), there is a bijection Φ defined for almost all $(x_1, x_2) \in \Delta_2$ that provides the following measure-theoretic isomorphism for suitable measures:

$$\begin{array}{ccc} \Delta_2 & \xrightarrow{T_{\mathrm{Brun}}} & \Delta_2 \\ \Big\downarrow \Phi & & \Big\downarrow \Phi \\ \mathcal{S}_{BR}^{\mathbb{N}} & \xrightarrow{S} & \mathcal{S}_{BR}^{\mathbb{N}} \end{array} \tag{6}$$

Theorem 20. *For almost all $\mathbf{t} \in \mathbb{T}^2$, there is $(x_1, x_2) \in \Delta_2$ such that the S-adic shift (X_γ, S) with $\gamma = \Phi(x_1, x_2)$ is measure-theoretically isomorphic to the translation by $\mathbf{t}$ on $\mathbb{T}^2$. Moreover, the words in X_γ form natural codings of the translation by $\mathbf{t}$, and the subpieces of the Rauzy fractal provide bounded remainder sets for this translation.*

References

[1] B. Adamczewski, Balances for fixed points of primitive substitutions, *Theoret. Comput. Sci.*, **307** (2003), 47–75.

[2] Adamczewski B., Symbolic discrepancy and self-similar dynamics, *Ann. Inst. Fourier (Grenoble)*, **54** (2004), 2201–2234.

[3] S. Akiyama, *Pisot numbers and greedy algorithm*, in Number theory (Eger, 1996), de Gruyter, 1998, 9–21.

[4] S. Akiyama, *Self affine tilings and Pisot numeration systems*, in Number Theory and its Applications (Kyoto 1997), K. Győry and S. Kanemitsu, eds., Kluwer Acad. Publ., 1999, 7–17.

[5] S. Akiyama, *On the boundary of self affine tilings generated by Pisot numbers*, J. Math. Soc. Japan, **54** (2002), 283–308.

[6] S. Akiyama, M. Barge, V. Berthé, J.Y. Lee, and A. Siegel, *On the Pisot substitution conjecture*, Mathematics of Aperiodic Order, Progress in Mathematics, vol. 309, Birkhäuser/Springer (Basel), 2015.

[7] P. Arnoux and G. Rauzy, Représentation géométrique de suites de complexité $2n+1$, *Bull. Soc. Math. France*, **119** (1991), 199–215.

[8] A. Avila and V. Delecroix, *Pisot property for the Brun and fully subtractive algorithms*, arXiv:1506.03692, preprint, 2015.

[9] A. Avila, P. Hubert, and A. Skripchenko, *Diffusion for chaotic plane sections of* 3*-periodic plane surfaces*, arXiv:1412.7913, preprint, 2014.

[10] V. Baladi and A. Nogueira, Lyapunov exponents for non-classical multidimensional continued fraction algorithms, *Nonlinearity*, **9** (1996), 1529–1546.

[11] P. R. Baldwin, A convergence exponent for multidimensional continued-fraction algorithms, *J. Statist. Phys.*, **66** (1992), 1507–1526.

[12] M. Barge, *Pure discrete spectrum for a class of one-dimensional substitution tiling systems*, arXiv:1403.7826, 2014.

[13] M. Barge and B. Diamond, *Coincidence for substitutions of Pisot type*, Bull. Soc. Math. France, **130** (2002), 619–626.

[14] M. Barge and J. Kwapisz, *Geometric theory of unimodular Pisot substitutions*, Amer. J. Math., **128** (2006), 1219–1282.

[15] V. Berthé, V. Delecroix, *Beyond substitutive dynamical systems: S-adic expansions*, RIMS Lecture note 'Kokyuroku Bessatu' **B46** (2014), 81–123.

[16] V. Berthé and M. Rigo (Eds.), Combinatorics, Automata and Number Theory, *Encyclopedia Math. Appl.*, vol. 135, Cambridge University Press, Cambridge, 2010.

[17] V. Berthé, J. Cassaigne, and W. Steiner, *Balance properties of Arnoux-Rauzy words*, Internat. J. Algebra Comput., **23** (2013), no. 4, 689–703.

[18] V. Berthé, S. Ferenczi, and L. Q. Zamboni, *Interactions between dynamics, arithmetics and combinatorics: the good, the bad, and the ugly*, in Algebraic and topological dynamics, vol. 385 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2005, pp. 333–364.

[19] V. Berthé, W. Steiner, and J. M. Thuswaldner, *Geometry, dynamics and arithmetic of S-adic shifts*, arXiv:1410.0331v1, preprint, 2014.

[20] V. Berthé, J. Bourdon, T. Jolivet and A. Siegel,, *A combinatorial approach to products of Pisot substitutions*, arXiv:1401.0704, Ergodic Theory Dynam. Systems, to appear.

[21] S. Bezuglyi, J. Kwiatkowski, K. Medynets and B. Solomyak, Finite rank Bratteli diagrams: structure of invariant measures, *Trans. Amer. Math. Soc.*, **365** (2013), 2637–2679.

[22] G. Birkhoff, Extensions of Jentzsch's theorem. *Trans. Amer. Math. Soc.*, **85** (1957), 219–227.

[23] P. Bougerol and J. Lacroix, Product of random matrices with applications to Schrödinger operators, Birkhaüser, 1985.

[24] A. J. Brentjes, *M*ultidimensional continued fraction algorithms, Mathematisch Centrum, Amsterdam, 1981.

[25] A. Broise-Alamichel, *On the characteristic exponents of the Jacobi-Perron algorithm*, Dynamical systems and Diophantine approximation, Sémin. Congr., vol. 19, Soc. Math. France, Paris, 2009, 151–171.

[26] A. Broise-Alamichel and Y. Guivarc'h, Exposants caractéristiques de l'algorithme de Jacobi-Perron et de la transformation associée, *Ann. Inst. Fourier (Grenoble)*, **5**1 (2001), 565–686.

[27] V. Brun, *Algorithmes euclidiens pour trois et quatre nombres*, Treizième congrès des mathèmaticiens scandinaves, tenu à Helsinki 18-23 août 1957, Mercators Tryckeri, Helsinki, 1958, pp. 45–64.

[28] J. Cassaigne, S. Ferenczi and L. Q. Zamboni, *Imbalances in Arnoux-Rauzy sequences*, Ann. Inst. Fourier (Grenoble), **50** (2000), 1265–1276.

[29] J. Cassaigne, S. Ferenczi, and A. Messaoudi, *Weak mixing and eigenvalues for Arnoux-Rauzy sequences*, Ann. Inst. Fourier (Grenoble), **58** (2008), 1983–2005.

[30] J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge University Press, New York, 1957.

[31] I. P. Cornfeld, S. V. Fomin and Y. G. Sinaĭ, *Ergodic theory*, Springer-Verlag, New York, 1982.

[32] J.-M. Dumont, and A. Thomas, Systèmes de numération et fonctions fractales relatifs aux substitutions, *Theoret. Comput. Sci.*, **65** (1989), 153–169.

[33] F. Durand, Linearly recurrent subshifts have a finite number of non-periodic subshift factors, *Ergodic Theory Dynam. Systems*, **20** (2000), 1061–1078. Corrigendum and addendum, *Ergodic Theory Dynam. Systems*, **23** (2003), 663–669.

[34] F. Durand, *Cobham's theorem and substitution subshifts*, this volume.

[35] S. Ferenczi, *Bounded remainder sets*, *Acta Arith.*, **61** (1992), 319–326.

[36] A. Furman, *Random walks on groups and random transformations*, Handbook of Dynamical Systems, Vol. 1A, 931–1014, North-Holland, Amsterdam, 2002.

[37] H. Furstenberg, Stationary processes and prediction theory, *Annals of Mathematics Studies*, vol. 44, Princeton University Press, Princeton, N.J., 1960.

[38] H. Furstenberg, Noncommuting random products, *Trans. Amer. Math. Soc.*, **108** (1963), 377–428.

[39] W. H. Gottschalk and G. A. Hedlund, *Topological dynamics*, American Mathematical Society Colloquium Publications **36**, American Mathematical Society, Providence, R. I. (1955).

[40] P. J. Grabner, P. Hellekalek, P. Liardet, *The dynamical point of view of low-discrepancy sequences*, Uniform Distribution Theory, **7** (2012), 11–70.

[41] M. Hollander and B. Solomyak, *Two-symbol Pisot substitutions have pure discrete spectrum*, Ergodic Theory Dynam. Systems, **23** (2003), 533–540.

[42] B. Host, *Valeurs propres des systèmes dynamiques définis par des substitutions de longueur variable*, Ergodic Theory Dynam. Systems, **6** (1986), 529–540.

[43] B. Host, *Représentation géométrique des substitutions sur 2 lettres.* Unpublished manuscript, 1992.

[44] S. Ito and H. Rao, *Atomic surfaces, tilings and coincidences I. Irreducible case*, Israel J. Math., **153** (2006), 129–156.

[45] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, Cambridge, 1995.

[46] M. Keane, Non-ergodic interval exchange transformations, *Israel J. Math.* **26** (1977), 188–196.

[47] J. C. Lagarias, The quality of the Diophantine approximations found by the Jacobi-Perron algorithm and related algorithms, *Monatsh. Math.*, **115** (1993), 299–328.

[48] M. Lothaire, *Algebraic Combinatorics on Words*, vol. 90, Encyclopedia of Mathematics and Its Applications. Cambridge University Press, 2002.

[49] M. Queffélec, *Substitution dynamical systems—spectral analysis*, vol. 1294 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, second ed., 2010.

[50] N. Priebe-Franck, *A primer on substitution tilings of euclidean space*, Expositiones Mathematicae, **26** (2008), 295–326.

[51] N. Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, vol. 1794 of Lecture Notes in Mathematics, Springer Verlag, 2002. Ed. by V. Berthé and S. Ferenczi and C. Mauduit and A. Siegel.

[52] G. Rauzy, *Nombres algébriques et substitutions*, Bull. Soc. Math. France, **110** (1982), 147–178.

[53] G. Rauzy, *Ensembles à restes bornés*, Seminar on number theory, 1983–1984 (Talence, 1983/1984), Univ. Bordeaux I, Talence, 1984, pp. Exp. No. 24, 12.

[54] M. Rédei, C. Werndl, *On the history of the isomorphism problem of dynamical systems with special regard to von Neumann's contribution*, Arch. Hist. Exact Sci., **66** (2012), 71–93.

[55] M. Rigo, Formal languages, automata and numeration systems, Volume 1: Introduction to combinatorics on words, Volume 2: Applications to recognizability and decidability. John Wiley & Sons, Inc., Hoboken, NJ, USA (2014).

[56] E. A. Robinson, Jr., *Symbolic dynamics and tilings of* $\mathbb{R}^d$, in Symbolic dynamics and its applications, vol. 60 of Proc. Sympos. Appl. Math., Amer. Math. Soc. Providence, RI, 2004, 81–119.

[57] K. Schmidt, *Algebraic coding of expansive group automorphisms and two-sided beta-shifts*, Monatsh. Math., **129** (2000), 37–61.

[58] F. Schweiger, *Multidimensional continued fractions*, Oxford Science Publications, Oxford University Press, Oxford, 2000.

[59] E. Seneta, *Non-negative matrices and Markov chains*, Springer Series in Statistics, Springer-Verlag, New York, 1981.

[60] A. Siegel and J. M. Thuswaldner, *Topological properties of Rauzy fractals*, Mém. Soc. Math. Fr. (N.S.) (2009), no. 118.

[61] B. Solomyak, *Dynamics of self-similar tilings*, Ergodic Theory Dynam. Systems **17** (1997), 695–738.

[62] W. Thurston, *Groups, tilings and finite state automata.* AMS Colloquium Lecture Notes, 1989.

[63] J. von Neumann, P. A. Halmos, *Operator methods in classical mechanics. II*, Ann. of Math. (2), **43** (1942), 332–350.

[64] P. Walters, *An introduction to ergodic theory*, Springer-Verlag, New York, 1982.

[65] A. Zorich, Deviation for interval exchange transformations, *Erg. Th. and Dyn. Syst.*, **17** (1997), 1477–1499.