INTERVAL EXCHANGE TRANSFORMATIONS

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Abstract. These are lecture notes for 4 introductory talks about interval exchange transformations and translation surfaces given by the author in Salta (Argentina) in November 2016.

Contents

In this course, we will see interval exchange transformations from different perspectives, namely:

(1) as a map of the interval,
(2) as a Poincaré map of a flow on a surface,
(3) as a symbolic dynamical system.

The aim of this course is to give an understanding of the interplay between these different point of views.

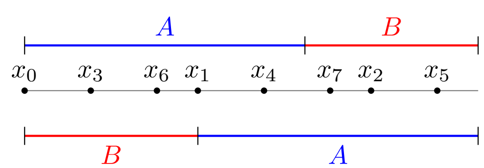
In the first course we will only consider rotations (or Sturmian languages) and will explore the link with continued fractions and $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. In the second course, we introduce the main actors: interval exchange transformations, translation surfaces, Rauzy induction and the $\mathrm{SL}(2,\mathbb{R})$-action. We will give a proof of Keane minimality condition. This second course can be seen as generalization of what was done in the first one.

In the third lecture we will introduce invariant measures and related concepts (linear recurrence and Boshernitzan condition). In the last lecture we discuss three deep results: Masur's asymptotic about counting, Kerckhoff-Masur-Smille theorem about generic unique ergodicity and a theorem about linear recurrence due to Kleinbock-Weiss and then improved by Chaika-Cheung-Masur.

1. Rotations and tori

1.1. **Rotation and their coding.** Let $\alpha \in (0,1)$ and let us consider the following map of the unit interval $T_\alpha : x \mapsto x + \alpha (mod\ 1)$. In other words

$$T_\alpha(x) = \begin{cases} x+\alpha & \text{if } x < 1-\alpha \\ x+\alpha-1 & \text{if } x > 1-\alpha \\ \text{undefined} & \text{if } x = 1-\alpha \end{cases}.$$



coding: $u = AABAABAB\ldots$

FIGURE 1. A picture of the rotation by $\alpha = (3-\sqrt{5})/2$ and the orbit of $x_0 = 0$ (here x_n denotes $T^n(x_0)$).

In dynamics, we are interested in the behavior of orbits under iteration. Namely, given an initial condition $x \in [0,1]$ how does look like the sequence $T(x)$, $T^2(x)$, $T^3(x)$, ...? Is it dense? Is it equidistributed?

One way to proceed, is to introduce a coding. The map T_α naturally induces a partition of the unit interval in two subintervals $I_A^{top} = [0, 1-\alpha)$ and $I_B^{top} = (1-\alpha, 1]$. Given an initial condition $x \in [0,1]$ we associates its **coding** which is the sequence $u = u_0 u_1 \ldots$ on $\{A, B\}$ defined by

$$u_n = \begin{cases} A & \text{if } T^n(x) \in I_A \\ B & \text{if } T^n(x) \in I_B \end{cases}.$$

As an example, the coding of the orbit $x = 0$ under T_α with $\alpha = (3-\sqrt{5})/2$ is

$$u = AABAABABAABAABABAABABAABAABABAABAA\ldots.$$

The **natural coding** or **language** of the map T_α is the set of finite words that appear in some coding. One can show that for the rotation of Figure 1 one has $L_\alpha = \{\varepsilon, A, B, AA, AB, BA, AAB, ABA, BAA, BAB, \ldots\}$. Given L_α and a non-negative integer n we denote by $L_{\alpha,n}$ the set of words of length n in L_α. Given a word $u = u_0 u_1 \ldots u_{n-1} \in L_\alpha$ we can associate the set of points in I whose orbit start with u, namely

$$I_u^{top} = I_0^{top} \cap T^{-1}(I_{u_1}^{top}) \cap T^{-2}(I_{u_2}^{top}) \cap \ldots \cap T^{-(n-1)}(I_{u_{n-1}}^{top}).$$

For each n, the sets $(I_u)_{u \in L_{\alpha,n}}$ form a partition of the interval (up to the extremity of these intervals).

We can also define a partition for T^{-1} given by $I_A^{bot} = [0, \alpha)$ and $I_B^{bot} = (\alpha, 1]$ and similarly, for $u = u_0 u_1 \ldots u_{n-1} \in L_\alpha$ the following interval

$$I_u^{bot} = I_{u_{n-1}}^{bot} \cap T(I_{u_{n-2}}^{bot}) \cap \ldots \cap T^{n-1}(I_{u_0}^{bot}).$$

By construction, T^n maps by translation I_u^{top} to I_u^{bot} (see Figure 2).

More generally, a **language** L is a non-empty set of words on a finite set called **alphabet** that:

- is **factorial**: if $u = u_0 u_1 \ldots u_{n-1}$ belongs to L then $u_1 \ldots u_{n-1}$ and $u_0 \ldots u_{n-2}$ belongs to L,
- is **prolongable**: for all $u \in L$ there exists a letter a such that $au \in L$ and a letter b so that $ub \in L$.

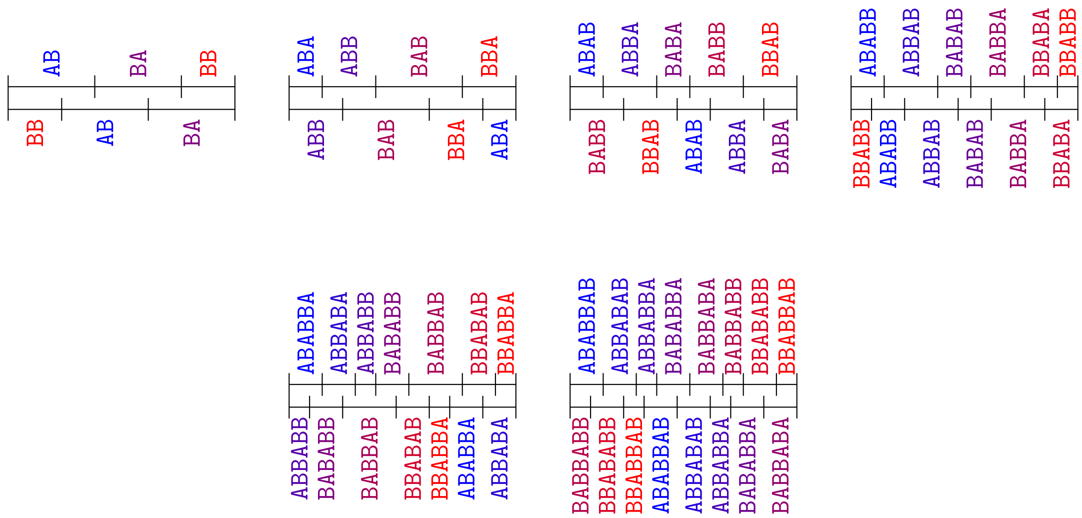


FIGURE 2. A picture for $n = 2, 3, 4, 5, 6, 7$ and $\alpha = (3 - \sqrt{5})/2$ of the partitions induced by $(I_u^{top})_{u \in L_{\alpha,n}}$ and $(I_u^{bot})_{u \in L_{\alpha,n}}$. The map T^n is a translation reduced to each of the subinterval I_u^{top} by either $\{n\alpha\}$ or $1 - \{n\alpha\}$.

The **complexity function** of a language L is the function p_L which to a non-negative integer associates the number of words of length n in L.

A language is called **uniformly recurrent** if for all positive integer n there exists an N so that any word of length N in L contains all words of length n as factors. This property is equivalent to the minimality (or density of orbits) of the underlying dynamical system (see Exercise 2).

A language L is said to be k-**balanced** [1] if for any pair of words $u, v \in L$ of the same length and any letter α we have $||u|_\alpha - |v|_\alpha| \leq k$. This property is related to invariant measures that will be discussed in Section 3.

Theorem 1. *Let L_α be the language of a rotation by an irrational number α. Then L_α*

(1) has complexity function $p(n) = n + 1$,
(2) is 1-balanced,
(3) is uniformly recurrent (in other words, all infinite orbits of T_α are dense).

Proof. The words of length n are exactly the number of intervals that T_α^n is made of. The limit points of these intervals are exactly $0, 1$ and the preimages $T^{-k}(1 - \alpha)$ for $k = 0, 1, \ldots, n$. As α is irrational, these preimages are all different and we hence obtain $n + 2$ different points that define $n + 1$ intervals.

By definition $T_\alpha^n(x) = \{x + n\alpha\}$ where $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. It is easily seen that the coding of x is given by

$$u_n = \begin{cases} A & \text{if } \lfloor x + (n+1)\alpha \rfloor - \lfloor x + n\alpha \rfloor = 0, \\ B & \text{if } \lfloor x + (n+1)\alpha \rfloor - \lfloor x + n\alpha \rfloor = 1. \end{cases}$$

Hence, for the coding u of x we have

$$|u_0 u_1 \ldots u_{n-1}|_B = \lfloor x + n\alpha \rfloor = \begin{cases} \lfloor n\alpha \rfloor & \text{if } x > 1 - \{n\alpha\} \\ \lfloor n\alpha \rfloor + 1 & \text{if } x < 1 - \{n\alpha\}. \end{cases}.$$

Hence the language is 1-balanced.

Uniform recurrence of the language L_α is equivalent to the fact that all infinite orbits of $T^n(\alpha)$ are dense (see Exercise 2). We can always build a sequence of integers $q_n \to \infty$ so that $\{q_n \alpha\} \to 0$ (one can use Dirichlet (or pigeonhole) principle). It follows that the sequence $\{m q_n \alpha\}_{n \geq 0, 1 \leq m < 1/\{q_n \alpha\}}$ is dense. So is the orbit of 0. Now to prove that every orbit is dense, it is enough to remark that $T^n(x) = \{x + n\alpha\} = \{T^n(0) + x\}$. $\square$

[1] Sometimes, author uses balanced for 1-balanced.

1.2. **Induced map, substitutions and continued fractions.** Given a dynamical system $T : X \to X$ and a subset $Y \subset X$ we can define the return time: $r(x) = r_Y(x) = \inf\{n > 0 : T^n(x) \in Y\}$. If it is well defined in Y we can define an *induced map* $T|_Y$ by setting for

$$T|_Y(x) = T^{r(x)}(x), \qquad \text{for } x \in Y.$$

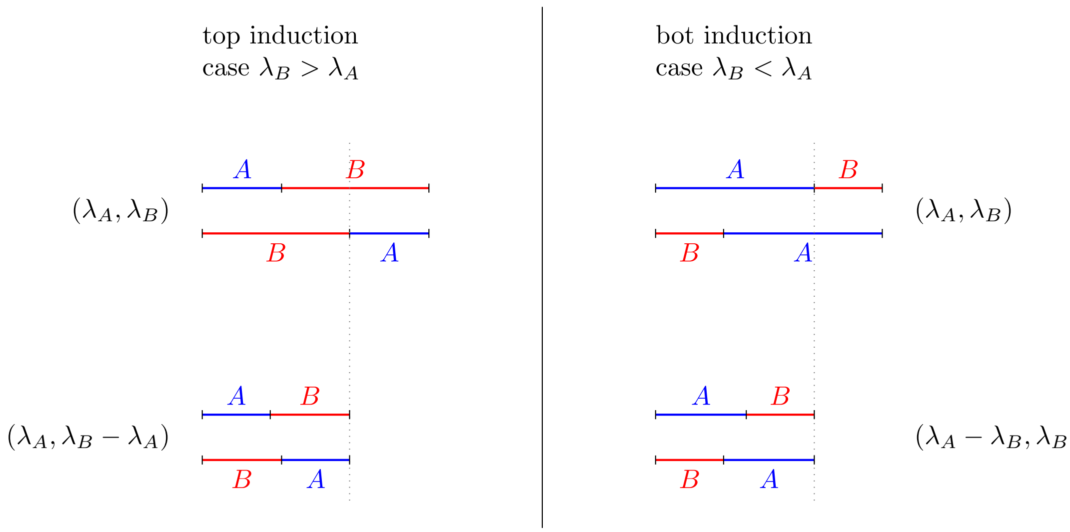
This is a very important notion to study dynamical system in general.

Let us consider a small generalization of rotations. Given $\lambda_A, \lambda_B \in \mathbb{R}_+$ let

$$T_\lambda : \begin{array}{ccl} [0, \lambda_A + \lambda_B] & \to & [0, \lambda_A + \lambda_B] \\ x & \mapsto & \left\{ \begin{array}{ll} x + \lambda_B & \text{if } x < \lambda_A \\ x - \lambda_A & \text{if } x > \lambda_A \end{array} \right. \end{array}$$

Rotations T_α corresponds to the case $\lambda_A = 1 - \alpha$ and $\lambda_B = \alpha$. Note that changing a scaling does not affect the dynamics: the maps T_λ and $T_{r\lambda}$ are conjugate for any $r > 0$.

The maps T_λ are parametrized by $\mathbb{R}_+^2$. We consider the following induction procedure: assume that $\lambda_A \neq \lambda_B$ and let $\lambda_{min} = \min(\lambda_A, \lambda_B)$; define $\mathcal{RV}(T_\lambda)$ to be the induced map by T_λ on $[0, |\lambda| - \lambda_{min}]$. This procedure is called the **Rauzy induction**.



It can easily be seen that the new map is just $T_{\lambda'}$ where

$$\lambda' = \left\{ \begin{array}{ll} (\lambda_A - \lambda_B, \lambda_B) & \text{if } \lambda_A > \lambda_B, \\ (\lambda_A, \lambda_B - \lambda_A) & \text{if } \lambda_B > \lambda_A. \end{array} \right.$$

Let us remark that one can deduce the coding of the points in T from the coding of the points in $T' = \mathcal{RV}(T)$: Let c be the coding of the point $x \in I'$ for T'. Then its coding for T is $\sigma^\varepsilon(c)$ where $\varepsilon \in \{top, bot\}$ is the type of the induction and where σ^ε are the following substitutions:

$$\sigma^{top} : \left\{ \begin{array}{lcl} A & \mapsto & AB \\ B & \mapsto & B \end{array} \right. \quad \text{and} \quad \sigma^{bot} : \left\{ \begin{array}{lcl} A & \mapsto & A \\ B & \mapsto & AB \end{array} \right. .$$

Proposition 2. *If λ_A and λ_B are not a rational multiple of each other, then the induction procedure $\mathcal{RV}$ can be applied infinitely often to T_λ. If λ_A and λ_B are rationally dependent, then after a certain number of steps we have $\lambda_A^{(n)} = \lambda_B^{(n)}$ where $\lambda^{(n)} = \mathcal{RV}^n(\lambda)$.*

Given λ with rationally independent coordinates, let $\varepsilon_0, \varepsilon_1, \ldots \in \{top, bot\}^{\mathbb{N}}$ be the succession of operations of the Rauzy Veech induction. Then the coding of the orbit of 0 by T_λ is

$$u = \lim_{n \to \infty} \sigma^{\varepsilon_0} \sigma^{\varepsilon_1} \ldots \sigma^{\varepsilon_n}(A)$$

If we factor $\varepsilon_0 \varepsilon_1 \ldots$ as $(top)^{a_0} (bot)^{a_1} (bot)^{a_2} \ldots$ then

$$\frac{\lambda_B}{\lambda_A} = a_0 + \cfrac{1}{a1 + \cfrac{1}{a_2 + \ldots}}.$$

The sequence $a_0, a_1, a_2, \ldots$ *is called the* continued fraction expansion *of* λ_B/λ_A.

Note that in the above result, only a_0 can be zero. All other terms are positive integers.

Example 1. The coding of the orbit of 0 in the golden rotation is the fixed point of the substitution $A \mapsto AAB, B \mapsto AB$.

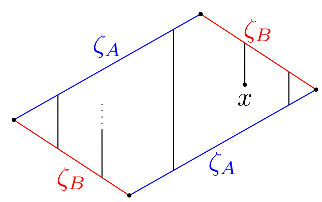
Proof. If λ_A and λ_B are rationally dependent then there exists N so that $N\lambda_A$ and $N\lambda_B$ are integers. Now, performing the Rauzy induction preserves the integer vectors and strictly decreases the norm. Hence after finitely many steps the vectors $(N\lambda_A, N\lambda_B)$ must land onto the line $\lambda_A = \lambda_B$ where the induction is not defined.

Now if λ_A and λ_B are rationally independent, then the Rauzy induction can be performed infinitely often. Indeed, if $\lambda_A^{(n)} = \lambda_B^{(n)}$, this expression is a linear expression in the components of λ since $\lambda = A_n \lambda^{(n)}$.

Let us now prove the second part of the statement. Let (λ_A, λ_B) be such that that $\lambda_B/\lambda_A \notin \mathbb{Q}$. Then, it is easily seen that $\lambda_A^{(n)} \to 0$ and $\lambda_B^{(n)} \to 0$. In particular, the only points that remain in the intersection of the domains of $T_\lambda^{(n)}$ is 0. Actually, the sequence A, $\sigma^{\varepsilon_0}(A)$, $\sigma^{\varepsilon_0}\sigma^{\varepsilon_1}(A)$, …is a sequence of finite words such that each term is a prolongation of the previous one. By construction, a given finite step describes the begining of the orbit of all points that belong to $I_A^{(n)}$. As the length of this interval goes to 0, the length of u is infinite and describes the coding of 0.

Writing λ_B/λ_A as a continued fraction directly follows from the definition of the algorithm. It can be proved by induction. ☐

1.3. **Suspensions.** A **flat torus** is the quotient of $\mathbb{R}^2$ by a lattice $\Lambda = \mathbb{Z}u \oplus \mathbb{Z}v$. The **linear flow in direction** θ in the torus $S = \mathbb{R}^2/\Lambda$ is the map ϕ_t^θ obtained by passing to the quotient the translation $x \in \mathbb{R}^2 \mapsto x + t(\cos(\theta), \sin(\theta))$. The vertical flow $\phi_t = \phi_t^{\pi/2}$ is simply called the linear flow. See Figure 3.



cutting sequence: $u = BAAB\ldots$

FIGURE 3. An orbit of the linear flow in a torus and its cutting sequence.

Actually rotations and linear flow on tori are basically the same objects: linear flows on tori are suspension flows of rotations. More formally, let T_λ be a rotation and τ_A, τ_B be two real numbers so that the area of the parallelogram determined by the vectors $\zeta_A = (\lambda_A, \tau_A)$ and $\zeta_B = (\lambda_B, \tau_B)$ is one. Performing the construction as in Figure 4, we obtain a torus $S_{\lambda,\tau}$ with an embedded interval such that the Poincaré map of the linear flow on the segment $[0, |\lambda|]$ is exactly the rotation T_λ. Moreover, the linear flow in the torus can be recovered as the suspension flow over T_λ with a roof function which is constant on the intervals I_A^{top} and I_B^{top} and respectively given by $-\tau_B$ and τ_A (see Figure 4).

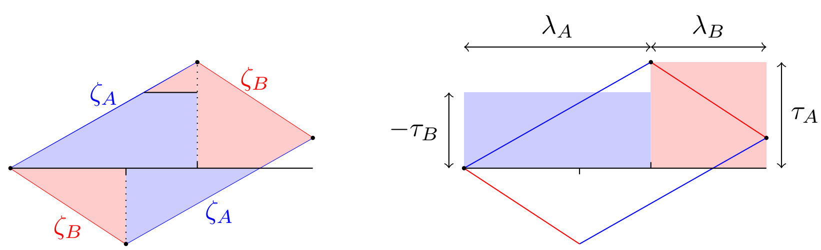


FIGURE 4. The torus in Figure 3 is actually a suspension flow of the rotation of Figure 1.

The set of area one bases in $\mathbb{R}^2$ can be identified with $\mathrm{SL}(2,\mathbb{R})$. Now two bases (u_1, v_1) and (u_2, v_2) define the same lattice, i.e. $\mathbb{Z}u_1 \oplus \mathbb{Z}v_1 = \mathbb{Z}u_2 \oplus \mathbb{Z}v_2$, if and only if there is a matrix M in $\mathrm{SL}(2,\mathbb{Z})$ so that $(u_1, v_1)M = (u_2, v_2)$. Hence the space of tori can be identified with the quotient $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SL}(2,\mathbb{Z})$.

The space $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SL}(2,\mathbb{Z})$ has a natural action by $\mathrm{SL}(2,\mathbb{R})$ on the left. On a basis (ζ_A, ζ_B), the group $\mathrm{SL}(2,\mathbb{R})$ simply acts as matrix on each of the vector on the left: $M \cdot (\zeta_A, \zeta_B) = (M\zeta_A, M\zeta_B)$. The flow induced by the action of $g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{SL}(2,\mathbb{Z})$ is called the **geodesic flow** or **Teichmüller flow**. Note that the action by the rotation matrix $r_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ corresponds to changing the direction of the linear flow: the vertical flow in $r_{-\theta}S$ is the flow in direction θ in S. In Figure 6, you can see the action of r_θ and g_t on a torus.

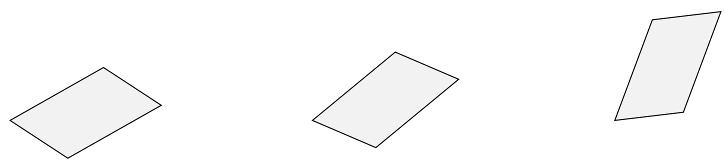


FIGURE 5. Two rotations $r_\theta S$ of some torus.

The following lemma shows that given a torus $\mathbb{R}^2/\Lambda$ one can always select a preferred basis. The proof is given as Exercise 13.

Lemma 3. *Let Λ be a lattice with no horizontal and no vertical vector. Then there exists a unique basis (ζ_a, ζ_b) of Λ such that*

(1) $0 < \mathrm{Re}(\zeta_a) < 1$ *and* $0 < \mathrm{Re}(\zeta_b) < 1$*,*
(2) $\mathrm{Im}(\zeta_a) > 0$ *and* $\mathrm{Im}(\zeta_b) < 0$*,*
(3) $\mathrm{Re}(\zeta_a + \zeta_b) \geq 1$*.*

The Rauzy induction also operates on suspensions. Given data $(\zeta_A = (\lambda_A, \tau_A)$ and $\zeta_B = (\lambda_B, \tau_B)$ we define

$$\mathcal{RV}(\zeta) = \begin{cases} (\zeta_A, \zeta_B - \zeta_A) & \text{if } \lambda_A < \lambda_B, \\ (\zeta_A - \zeta_B, \zeta_B) & \text{if } \lambda_A > \lambda_B. \end{cases}$$

The map $\mathcal{RV}$ on the set of suspension data is a linear map and is called the **Rauzy-Veech induction**. More precisely, it corresponds to a left action of $\mathrm{SL}(2,\mathbb{Z})$ by one of the following elementary matrices $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. See Figure 6.

The **normalized Rauzy induction** is the map $\lambda \mapsto \mathcal{RV}(\lambda)/|\mathcal{RV}(\lambda)|$ acting on the set of $\lambda = (\lambda_A, \lambda_B)$ so that $\lambda_A + \lambda_B = 1$. We also define a **normalized Rauzy-Veech induction** given by $\zeta \mapsto g_{-\log \mathcal{RV}(\lambda)} RV(\zeta)$. It acts on the set P of bases (ζ_A, ζ_B) so that $\mathrm{Re}(\zeta_A + \zeta_B) = 1$. By Lemma 3 we have the following proposition.

Proposition 4. *Let $P \subset \mathrm{SL}(2,\mathbb{R})/\,\mathrm{SL}(2,\mathbb{Z})$ be the set of tori with no horizontal and no vertical vectors and such that there canonical basis (ζ_A, ζ_B) is so that $\mathrm{Re}(\zeta_A + \zeta_B) = 1$.*

Then, the first return map of the Teichmüller flow on P is exactly the normalized Rauzy-Veech induction.

1.4. **Further results.** Actually, there is something miraculous with Sturmian sequences. The properties they have are also a characterization!

Theorem 5. *Let $L \subset \{A, B\}^*$ be a (factorial prolongable) uniformly recurrent language. Then the following are equivalent*

(1) L is a coding of an irrational rotation,
(2) $p_L(n) = n + 1$,
(3) L is 1-balanced,
(4) any u in L has exactly two return words.

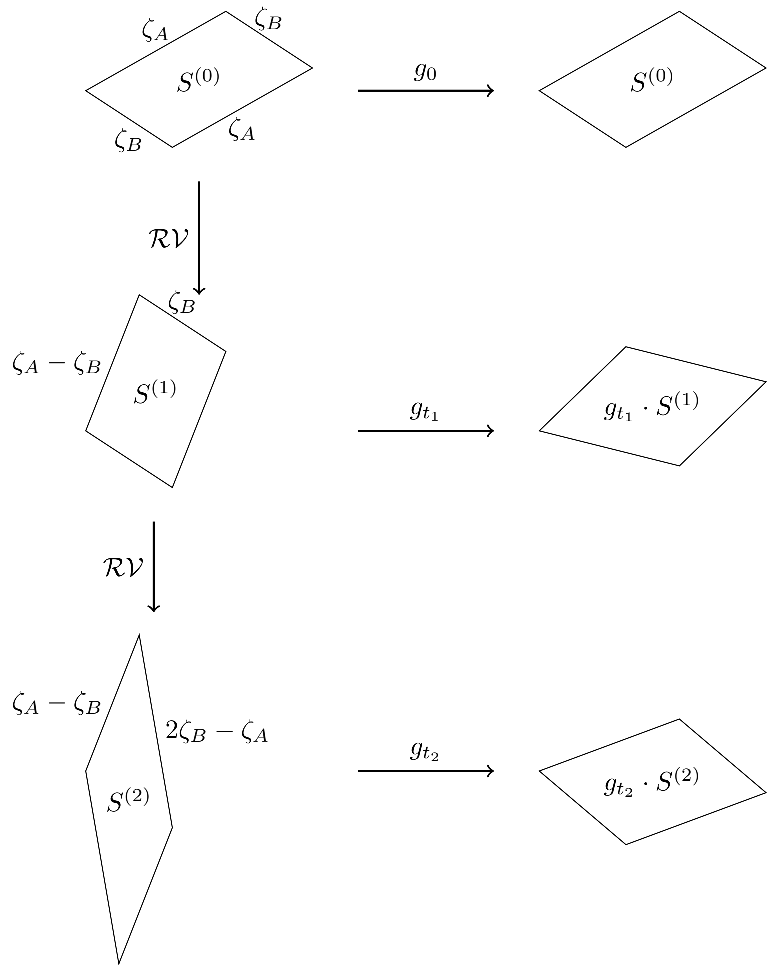


FIGURE 6. The left $\mathrm{SL}(2,\mathbb{R})$-action (via the Teichmüller flow) and the right $\mathrm{SL}(2,\mathbb{Z})$-action (via the Rauzy-Veech induction) on a torus.

For the proofs you can read the original paper of Morse and Hedlund [MH40], the chapter "Sturmian sequences" by P. Arnoux in [Fog02] and the article of L. Vuillon [Vui01]. There are also other characterizations based on the number of palindroms (X. Droubay et G. Pirillo [DP99]) and using palindromic closure (A. de Luca [dL97]).

There are also other dynamical contexts where Sturmian shifts appear: maximizing measures ([Bou00], [Jen08]) and complex dynamics.

2. Interval exchange transformations and translation surfaces

One possible generalization of rotations are interval exchange transformations. An interval exchange transformation is a piecewise transformation of an interval with a finite number of pieces. The rotation corresponds to the case of two intervals.

For a more detailed introduction I recommand the two following references:

- Masur-Tabachnikov [MT02]: the text is written with an emphasis on translation surfaces rather than interval exchange transformations. The first chapter contains a lot of examples and there are examples of non-uniquely ergodic billiards as well as a complete proof of Kerckhoff-Masur-Smillie theorem.
- Yoccoz [Yoc06]: the point of view in this text is more toward interval exchange transformations. There is the construction of the invariant measure of the Rauzy-Veech induction. It has a more combinatorial flavour than the other text.

2.1. **Interval exchange transformations.** Let A be a finite alphabet of cardinality d and consider a couple of bijections π^{top} and π^{bot} from A to $\{1, 2, \ldots, d\}$. These bijections can be thought as two different linear orders on A.

Let $\lambda = (\lambda_i)_{i \in A}$ be a vector of positive real numbers. Then set for $k = 0, 1, \ldots, d$ the following quantities

$$\alpha_k^{top} = \sum_{i:\ \pi^{top}(i) \leq k} \lambda_i \quad \text{and} \quad \alpha_k^{bot} = \sum_{i:\ \pi^{bot}(i) \leq k} \lambda_i.$$

The points α_k^{top} (respectively α_k^{bot}) determines two partitions of the interval $[0, |\lambda|]$. Namely for $i \in A$ set

$$I_i^{top} = (\alpha^{top}_{\pi^{top}(i)-1},\ \alpha^{top}_{\pi^{top}(i)}) \quad \text{and} \quad I_i^{bot} = (\alpha^{bot}_{\pi^{bot}(i)-1},\ \alpha^{bot}_{\pi^{bot}(i)}).$$

The **interval exchange transformation** defined by the data π and λ is the map $T : [0, |\lambda|] \to [0, |\lambda|]$ that for each $i \in A$ is a translation from I_i^{top} onto I_i^{bot}. See Figure 7.

The permutation π is called **irreducible** (or **indecomposable**) if there is no k with $1 \leq k < d$ so that $(\pi^{top})^{-1}(\{1, 2, \ldots, k\}) = (\pi^{bot})^{-1}(\{1, 2, \ldots, k\})$. An interval exchange transformation with a reducible permutation decomposes as two independent interval exchange transformations.

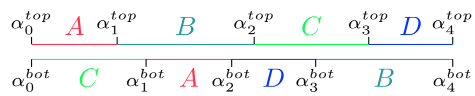


FIGURE 7. Picture of an interval exchange transformation. The interval on top is cut according to the partition (I_i^{top}) while the interval in the bottom is according to (I_i^{bot}).

The points $\alpha_1^{top}, \alpha_2^{top}, \ldots, \alpha_{d-1}^{top}$ (respectively $\alpha_1^{bot}, \alpha_2^{bot}, \ldots, \alpha_{d-1}^{bot}$) are called the **top singularities** (resp. the **bottom singularities**) of the interval exchange transformation.

2.2. **Rauzy induction.** We now introduce Rauzy induction which is a natural generalization to what we did with rotations. It first appeared in a paper of Gérard Rauzy [Rau79].

Starting from a d-interval exchange transformation T with data (π, λ) with π irreducible, we consider the two right most intervals with labels $(\pi^{top})^{-1}(d)$ and $(\pi^{bot})^{-1}(d)$. Then the **Rauzy induction** consists of inducing T on $[0, |\lambda| - \min(\lambda_{\alpha^{top}_{d-1}}, \lambda_{\alpha^{bot}_{d-1}})]$. In the exceptional case where $\lambda_{\alpha^{top}_{d-1}} = \lambda_{\alpha^{bot}_{d-1}}$ the induction is not defined. The label of the longest interval is called the **winner** and the shortest the **loser**. See Figure 8 for a picture of the Rauzy induction.

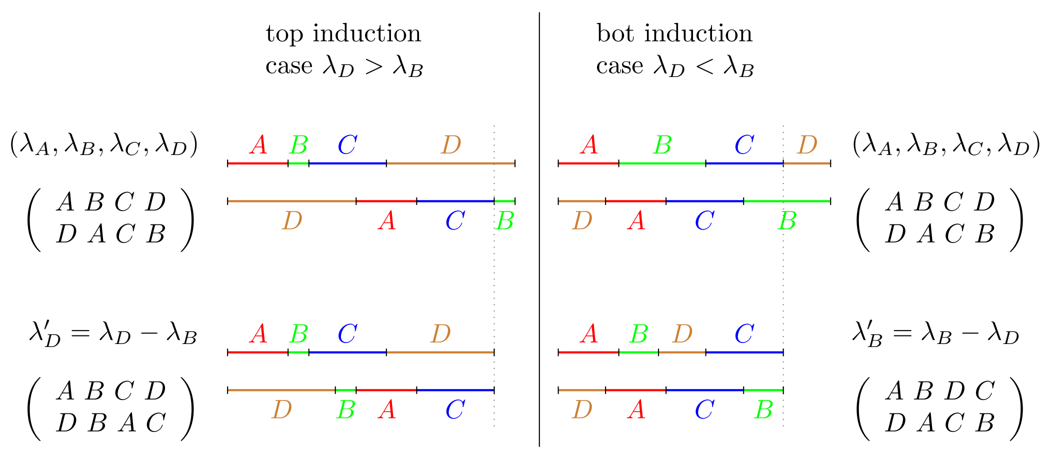


FIGURE 8. The two cases of the Rauzy induction for interval exchange transformation.

One difference with rotations, is that now the permutation data π might change. But the operation is purely combinatorial and only depends on whether $\alpha_{d-1}^{top} > \alpha_{d-1}^{bot}$ (top induction) or $\alpha_{d-1}^{bot} > \alpha_{d-1}^{top}$ (bottom induction). The set of permutations π that are obtained from these two operations are called **Rauzy classes**.

They form the vertices of an oriented graph whose edges correspond to the top and bottom inductions which is called a **Rauzy diagram**. With Exercises 6 and 7 you will get more familiar with irreducible permutations and Rauzy diagrams.

$$\begin{tikzcd}[column sep=large]
\begin{pmatrix} A & B & C \\ C & A & B \end{pmatrix} \arrow[loop left, "\substack{\textbf{bot}\\ win=B\\ los=C}"] \arrow[r, shift left, "\substack{\textbf{top}\\ win=C\\ los=B}"] & \begin{pmatrix} A & B & C \\ C & B & A \end{pmatrix} \arrow[l, shift left, "\substack{\textbf{top}\\ win=C\\ los=A}"] \arrow[r, shift left, "\substack{\textbf{bot}\\ win=A\\ los=C}"] & \begin{pmatrix} A & C & B \\ C & B & A \end{pmatrix} \arrow[l, shift left, "\substack{\textbf{bot}\\ win=A\\ los=B}"] \arrow[loop right, "\substack{\textbf{top}\\ win=B\\ los=A}"]
\end{tikzcd}$$

Figure 9. The Rauzy diagram for 3-iet.

If we iterate the Rauzy induction from a given interval exchange transformation $(\pi, \lambda) = (\pi^{(0)}, \lambda^{(0)})$ we obtain a sequence of interval exchange transformations $(\pi^{(1)}, \lambda^{(1)})$, $(\pi^{(2)}, \lambda^{(2)})$, ... and the permutations form a path in the Rauzy diagram.

The action of the Rauzy induction on lengths $\lambda \mapsto \lambda'$ is given by a linear transformation. We denote by $A(\pi, \lambda)$ the matrix so that $\lambda = A(\pi, \lambda)\lambda'$. The matrix $A(\pi, \lambda)$ is an elementary matrix (i.e. there are 1 on the diagonal and at one more extra place). Given $m \leq n$ we also define

$$A_{m,n}(\pi, \lambda) = A(\pi^{(m)}, \lambda^{(m)})\ A(\pi^{(m+1)}, \lambda^{(m+1)})\ \ldots\ A(\pi^{(n-1)}, \lambda^{(n-1)}).$$

These matrices satisfy $A_{m,n}(\pi, \lambda)\lambda^{(n)} = \lambda^{(m)}$. In particular, if the Rauzy induction is a periodic path of period p, then the vector λ is a Perron eigenvector of $A_{0,p}$. You can have a look at Exercise 8 for a concrete example.

The rows of $A_{m,n}$ describe the composition of Rohlin towers of the n-th level in terms of the m-th one.

2.3. **Keane theorem.** We now study the first important dynamical property of interval exchange transformations: their minimality. This result is due to Michael Keane [Kea75]. The proof of the result can also be found in the survey [MT02] (Theorem 1.8) and [Yoc06] (Section 3).

Let $T = T_{\pi,\lambda}$ be a d-interval exchange transformations. A **connection** for T is a triple (m, α, β) so that α is a singularity of T, β is a singularity of T^{-1} and $T^m\beta = \alpha$. In other words, there is a point in the interval such that both its orbit in the future and in the past are finite.

Theorem 6. *Let $T = T_{\pi,\lambda}$ be a d-interval exchange transformation with π irreducible. Then the following are equivalent*

(1) T has no connection,
(2) the complexity of the natural coding of T is $p(n) = (d-1)n + 1$,
(3) the Rauzy induction is well defined for all times.

If these conditions are satisfied then

- *the iterates of the Rauzy-Veech induction $(\pi^{(n)}, \lambda^{(n)})$ are such that $\lambda^{(n)} \to 0$,*
- *each letter in the alphabet wins and loses infinitely often.*

Proof. Let us first prove that (1) is equivalent to (2). Let T be a d-interval exchange transformations. The number of intervals of T^n are obtained from the one of T^{n-1} and the points $T^{-n}\alpha_i^{top}$ with $i = 1, \ldots, d-1$. Hence, exactly $d-1$ intervals of T^n are cut into two intervals, unless there is a connection in which case there are less.

If the Rauzy induction stops after m steps, then it is because at this step the rightmost intervals of top and bottom have the same length. In other words $T^{(m)}$ has a connection of length 1. Because, $T^{(m)}$ is an induced map, it lifts to a connection of T (which might be longer than 1). We proved that (1) implies (3).

Before proving the converse, we prove the second part of the theorem. Let $T = T_{\pi,\lambda}$ be an interval exchange transformation with no connection. It is easy to notice that $\lambda^{(n)} \to 0$ if and only if every letters win infinitely often. But in Rauzy induction, after a letter wins for some time it loses. So this condition is also equivalent to the fact that every letter lose infinitely often. Now assume that $\lambda^{(n)} \not\to 0$. Let $A' \subset A$ be the subset of letters that win only finitely many times. Let $(\pi^{(m)}, \lambda^{(m)})$ be a step of the Rauzy induction so that after this step no letter in A' wins. Then, all letters of A' must be on the left part of both $\pi^{(m),top}$ and $\pi^{(m),bot}$. In other words, $\pi^{(m)}$ is reducible. Which contradicts that π was reducible as Rauzy induction preserves irreducibility.

Now we prove that (3) implies (1). Let T be an interval exchange transformation with a connection (m, α, β). Let $x = \min\{\beta, T\beta, \ldots, T^m\beta\}$. Assume that the Rauzy induction is well defined. As we saw above, $\lambda^{(n)} \to 0$. Hence there is a time m so that $|\lambda^{(m)}| > x$ but $|\lambda^{(m+1)}| < x$. The only possibility for the m-th step is that x is the singularity of the rightmost interval in both the top and bottom interval. Which contradicts the fact that the Rauzy induction was well defined. □

Theorem 7. *Let T be an interval exchange transformation on $d \geq 2$ intervals. If T has no connection then it is minimal (i.e. all infinite orbits are dense).*

Proof. We first claim that if T has a periodic orbit then it has a connection. Indeed, periodic orbits comes into families: around a periodic orbit there is a periodic interval. At the boundary of this interval, there is necessary a connection.

Now assume that T is an interval exchange without connection and let J be a subinterval. Consider the induced map T_J of T on J. By Poincaré recurrence theorem it is defined almost everywhere, and where it is defined, it is locally a translation. The singularity of this map are exactly the pull-back of the singularities of T and the extremities of J. There are hence at most $d+2$ of them. Hence T_J is an interval exchange transformation on at most $d+2$ intervals.

For each of the subinterval J_i of T_J let r_i be the return time. By definition,

$$\widetilde{J} = \bigcup_{i=1}^{d'} \bigcup_{n=0}^{r_i - 1} T^n(J_i)$$

is a T-invariant set.

Assume by contradiction that $\widetilde{J}$ is not the whole interval I. Then there is a boundary point x between $\widetilde{J}$ and $I \backslash \widetilde{J}$. When iterating this point (either in the past or in the future) it either becomes a singularity or remains a boundary point between $\widetilde{J}$ and $I \backslash \widetilde{J}$. As there are finitely many such points, if the orbit is defined it is periodic. In either case, we found a connection. □

Proposition 8. *The set of λ that does not satisfy the hypothesis of Theorem 7 is contained in a countable union of rational hyperplanes (i.e. of the form $\{(x_1, x_2, \ldots, x_d) : \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_d x_d = 0\}$ for some $(\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Q}^2$).*

In particular, if π is irreducible and the coordinates $\lambda_1/\lambda_d, \ldots, \lambda_{d-1}/\lambda_d$ are rationally independent then $T_{\pi,\lambda}$ is minimal.

Proof. By Theorem 6, the presence of saddle connections is detected through Rauzy induction. Namely, there is a saddle connection if and only if after some step of induction the rightmost intervals in top and bottom have the same length. But each of these conditions is linear in λ with integer coefficients because we have $\lambda = A_{0,n}(\pi, \lambda)\lambda^{(n)}$. □

2.4. **Translation surfaces and suspensions of iet.** Translation surfaces are generalization of tori that we saw in Section 1.3. A translation surface is a surface obtained by gluing finitely many polygons where the edges are identified through translation. Torus can be constructed from parallelograms with identified opposite side. The vertices in the polygon plays a special role and we will call them the *vertices* of the surface.

We first see how to construct a translation surface from an interval exchange transformation. A **suspension data** for the interval exchange transformation $T_{\pi,\lambda}$ is a vector $\tau \in \mathbb{R}^A$ such that

$$\forall 0 \leq k \leq d-1, \qquad \sum_{\pi^{top}(i) \leq k} \tau_k > 0 \qquad \text{and} \qquad \sum_{\pi^{bot}(i) \leq k} \tau_k < 0.$$

Given such vector τ one can build a *translation surface* $S_{\pi,\lambda,\tau}$ and an interval $I \subset S$ so that the translation flow on S is a suspension of T and so that the Poincaré map induced on I is exactly T. See Figure 10.

As in the case of the torus, there is a natural relationship. The **translation flow** ϕ_t in a translation surface S is the flow which is defined locally in each polygon defining the surface by setting $\phi_t(x) = x + it$.

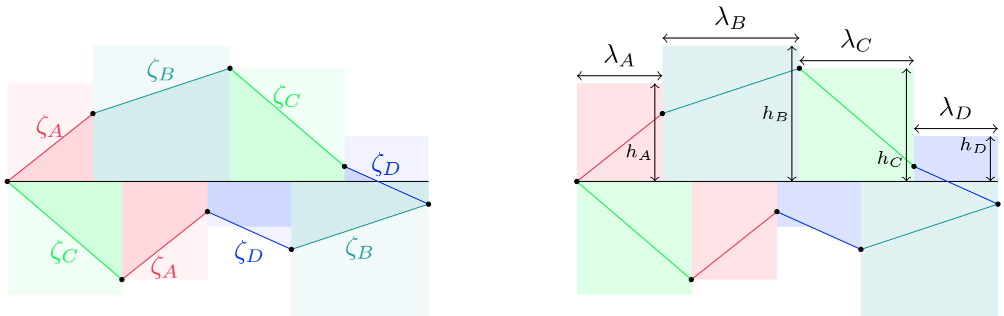


FIGURE 10. A suspension of the interval exchange transformation of Figure 7.

As the edges are glued by translation this gives a well defined flow on the whole surface *except at the vertices*. We can also define the translation flow in direction θ by setting $\phi_t^\theta(x) = x + t(\cos(\theta), \sin(\theta))$. In a suspension $S_{\pi,\lambda,\tau}$, the interval exchange transformation is a Poincaré map of the translation flow.

There is a natural equivalent of connections for translation surfaces. A **saddle connection** in a translation surface S is a straight line segment that joins two vertices of S. In other words there exists a point x_0, a direction θ_0 and two times $t_0 < t_1$ so that both $\phi_{t_0}^{\theta_0}(x_0)$ and $\phi_{t_1}^{\theta_0}(x_0)$ are vertices of the surface. As an example, the sides of the polygons are saddle connections. As well as any diagonals in it.

The suspension method actually allows to build most translation surfaces.

Proposition 9 (Veech)**.** *Let S be a translation surface with no vertical saddle connection. Then there is an horizontal interval I in S so that S can be obtained as a suspension data from I.*

We will not provide the proof here.

As a corollary of Keane theorem we have

Corollary 10. *The set of non-minimal direction in a translation surface is at most countable.*

2.5. **Strata, Teichmüller flow and Rauzy-Veech induction.** Recall that the space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ was a way to describe the set of tori. We now turn to a similar definition in the case of surfaces.

Given a translation surfaces, the vertices of the surfaces may have more angle then 2π around them. For example, in Figure 10 all the vertices of the polygon define only one vertex in the surface. And it has an angle 6π. More generally, a vertex of **degree** is a vertex of angle $2(1+\kappa)\pi$.

Let $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_m)$ be a partition of an even integer with possibly some part being 0. The **stratum of translation surfaces** $\mathcal{H}(\kappa)$ is the set of equivalence classes of translation surfaces with conical angles κ. Two surfaces are identified if we can pass from one to the other by cut and paste operations (see Figure 11). We denote $\mathcal{H}^1(\kappa)$ the subset of $\mathcal{H}(\kappa)$ made of surfaces of area one. For example $\mathcal{H}(0) = \mathrm{GL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ and $\mathcal{H}^1(0) = \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. For strata different from $\mathcal{H}(0)$ the topology is more complicated.

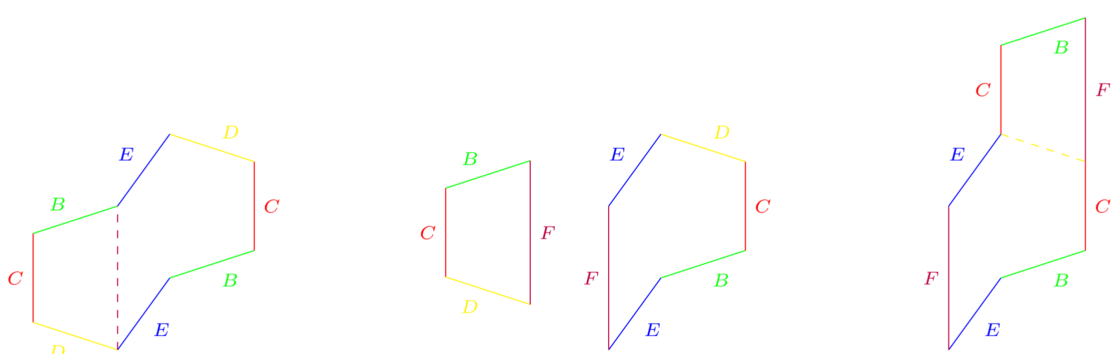


FIGURE 11. Cut and paste operations in a surface.

The group $\mathrm{SL}(2,\mathbb{R})$ acts on translation surfaces by their linear action on the defining polygons. Note that this action preserves the profile of singularities and the area. So that we get an action of $\mathrm{SL}(2,\mathbb{R})$ on each stratum of area one surfaces $\mathcal{H}^1(\kappa)$. The **Teichmüller flow** is the action of the diagonal flow $\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ on the strata $\mathcal{H}^1(\kappa)$.

As in the torus case, the Rauzy induction extends to suspension. This is a particular case of a cut and paste operation and hence while applying a step of the Rauzy-Veech induction the surface belongs to the same equivalence class of surfaces in the stratum!

Here is an elementary proposition that emphasize the link between the (normalized) Rauzy-Veech induction and the Teichmüller flow.

Proposition 11. *The normalized Rauzy-Veech induction is the first return map of the Teichmüller flow on the strata of translation surface with a marked outgoing horizontal separatrix.*

More precisely, if π is an irreducible permutation on r letters and let $\alpha^{top} = (\pi^{top})^{-1}(d)$ and $\alpha^{bot} = (\pi^{bot})^{-1}(d)$. Then the roof function over (π, λ) is given by

$$\log\left(\frac{1}{1 - \min(\lambda_{\alpha^{top}}, \lambda_{\alpha^{bot}})}\right).$$

2.6. **Best approximations.** In this section we introduce the best approximations. As we will see in the next section, they are very useful to control dynamical properties of an interval exchange transformation.

The **holonomy** (or **displacement vector**) of a saddle connection γ is the vector in $\mathbb{R}^2$ that is obtained by developing the saddle connection (see Figure 12). We will denote by $V(S)$ the set of holonomies of a translation surface S. In Exercise 17 you will compute some holonomies.

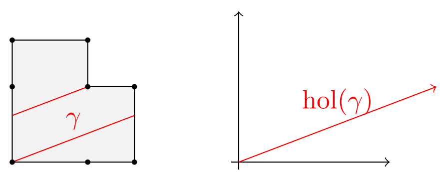


FIGURE 12. The holonomy of a saddle connection

In the case of the torus, the set of saddle connections of the torus $\mathbb{R}^2/\Lambda$ identifies with the primitive points of the lattice. That is, the set of vectors $v \in \Lambda \backslash \{0\}$ so that $\Lambda \cap (\mathbb{R}v) = \mathbb{Z}v$. In the case of the square lattice $\mathbb{Z}^2$, these are just the vectors (m, n) with $\gcd(m, n) = 1$.

Among the saddle connections, some of them will play a particular role.

Definition 12. Let S be a translation surface. A saddle connection γ is a **best approximation** if its holonomy has positive imaginary part and there is an immersed rectangle in S so that its diagonal is γ.

In some sense, best approximations are the saddle connections that approximate well the linear flow. Note that the definition of saddle connection is independent of the vertical direction. However, best approximations do changes as we vary the direction.

In exercise 14 you will show that in the case of the torus, the best approximations are easily obtained from the Rauzy-Veech induction.

As we will see later on, many information on best approximations give you information on the dynamics of the linear flow. Keane theorem can already be seen as an example of that (a vertical saddle connection is indeed a best approximation).

2.7. **Notes and further results.** We already saw that Sturmian languages can be characterized in a lot of different ways. The languages of k-interval exchange transformations are not so nice to describe. However there is a characterization due to [FZ08] and [BC] expressed in terms of bifurcation of bispecial words.

3. EQUIDISTRIBUTION

A property that can be seen as a refinement of minimality is the one of equidistribution, i.e. whether the orbit fills all part of the interval equally. There are various level of equidistribution: linear recurrence, unique ergodicity, ergodicity. All these concepts are closely related to invariant measures.

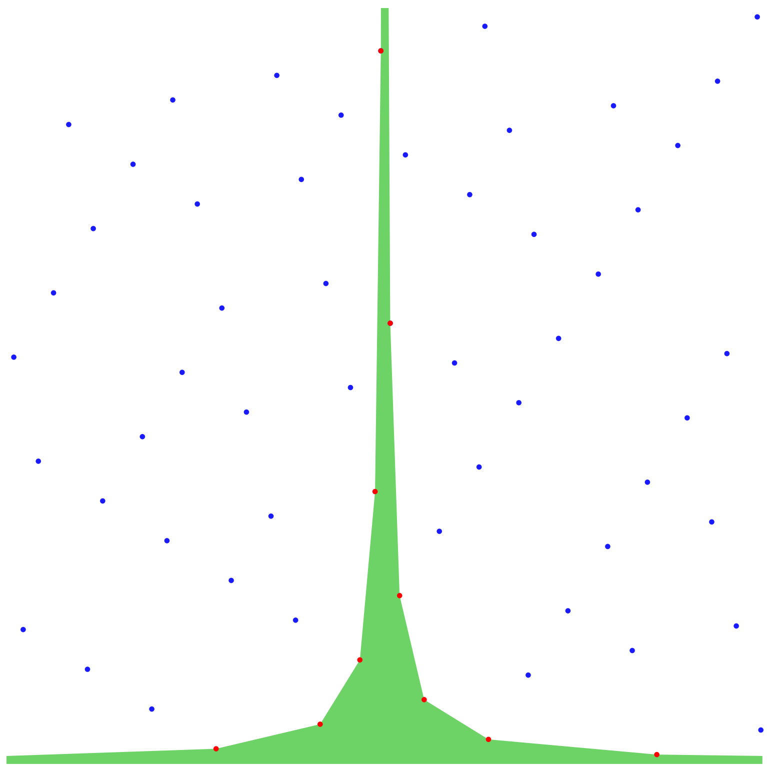


FIGURE 13. The primitive vectors of the square lattice rotated in the direction of coslope $\frac{1+\sqrt{5}}{2}$. The shaded zone is obtained from the broken lines joining the holonomies of best approximations.

3.1. **Crash course in ergodic theory.** Ergodic theory is the study of dynamical system through measure theory (where in continuous dynamical system one would use topology). We recall some results that will be used later on. For a general reference on ergodic theory one can consult the chapter by S. Ferenczi and T. Monteil [FM10] (which focuses on symbolic dynamics) or the book by A. Katok and B. Hasselblatt [KH95] (which deals in a more abstract setting).

In Exercises 9, 10 and 11 you will see links between measurable and topological dynamics.

Let $(X, \mathcal{B})$ be a measure space and $T : X \to X$ a measurable transformation. A probability measure μ is **invariant** for T if $T^*\mu = \mu$. In other words, if for all measurable sets $A \subset X$ we have $\mu(A) = \mu(T^{-1}A)$.

For example, any interval exchange transformation preserves the Lebesgue measure of the interval.

Having an invariant probability measure implies some strong properties as the following:

Theorem 13 (Poincaré recurrence)**.** *Let (X, T, μ) a dynamical system with invariant (probability) measure μ. Then for all measurable set $A \subset X$, for μ-almost every $x \in A$ there exists an n so that $T^n x \in A$.*

Using the compacity of the space of measure, one can show that continuous maps always admit invariant measures.

Proposition 14 (Krylov-Bogolyubov theorem)**.** *Let X be a compact metric space and $T : X \to X$ a continuous map. Then there exists an invariant (probability) measure μ for T.*

An invariant measure μ of a measurable map $T : X \to X$ is called *ergodic* if for any invariant set A (i.e. $T^{-1}A = A$) we have either $\mu(A) = 0$ or $\mu(A) = 1$. In other words, there is no measurable way to decompose the space into two subspaces. This notion is close to the minimality but in a measurable context.

In the case of continuous map, the set of invariant measures has some structure.

Proposition 15. *Let $T : X \to X$ be a continuous map. Then the set $\mathcal{M}_T$ of ergodic measures is a Choquet simplex. The extremal points of T are the ergodic measures of T.*

Moreover, the ergodic measures are mutually singular.

Interval exchange maps are not continuous. Though, one can compactify them by doubling the orbits of singularities. We will see in the next section that interval exchange transformations admit only finitely many ergodic measures.

An important result of ergodic theory is Birkhoff theorem that asserts that ergodicity is enough to ensure equidistribution of almost every orbit.

Theorem 16 (Birkhoff Theorem). *Let (X, T, μ) be a dynamical system with an invariant ergodic measure μ. Then for any integrable function $f : X \to \mathbb{R}$, we have that the Birkhoff sums $S_n(f, x) = f(x) + f(T^x) + \ldots + f(T^{n-1}x)$ are such that $S_n(f, .)/n$ converges to $\int_X f d\mu$ almost everywhere and in L^1.*

Note that S_n is a time average whereas $\int f$ is a space average.

A continuous dynamical system is called **uniquely ergodic** if it admits only one invariant measure.

Proposition 17. *If (X, T) is a continuous dynamical system which is uniquely ergodic then the invariant measure μ is ergodic. Moreover, for any continuous function f, the Birkhoff averages $S_n(f, .)/n$ converges uniformly to $\int_X f d\mu$.*

3.2. **Invariant measures of interval exchange transformations.** In this section, we show that the invariant measures of interval exchange transformations are naturally seen on some simplex obtained from the Rauzy induction.

Let $T = T_{\pi,\lambda}$ be an interval exchange transformation. Let $A_n(\pi, \lambda)$ be the matrix associated to the Rauzy induction which satisfies $\lambda = A_n(\pi, \lambda)\lambda^{(n)}$. We associate to (π, λ) the following cone

$$C(\pi, \lambda) = \bigcap_{n \geq 0} A_n(\pi, \lambda)\mathbb{R}_+^A.$$

Note that the cones $A_n(\pi, \lambda)\mathbb{R}_+^A$ are nested and the vectors λ' in $A_n(\pi, \lambda)\mathbb{R}_+^A$ are exactly the ones for which the n-th first steps of the Rauzy induction of (π, λ') coincide with the one of (π, λ). Hence, the vectors in $C(\pi, \lambda)$ are exactly the length data of interval exchange transformations $T_{\pi,\lambda'}$ for which the Rauzy induction follows the same (infinite) path in the Rauzy diagram as (π, λ).

Let us denote by $\mathcal{M}_T$ the set of Borelian invariant measures of T. We have a natural map from $\mathcal{M}_T$ to $\mathbb{P}\mathbb{R}_+^A$ given by $\mu \mapsto (\mu(I_i))_{i \in A}$. This map is linear.

Proposition 18. *Let $T = T_{\pi,\lambda}$ be an interval exchange transformations without connections. Then the map $\mathcal{M}_T \to \mathbb{P}\mathbb{R}_+^A$ defined above is an homeomorphism onto $\mathbb{P}C(\pi, \lambda)$.*

Proof. Since T has dense orbits, the invariant measures have no atom. Given an invariant measure μ of T we can associate an increasing homeomorphism H_μ of $[0, 1]$ by $H_\mu(x) = \mu([0, x])$.

Let $T_\mu = H_\mu \circ T \circ (H_\mu)^{-1}$. We claim that T_μ is an interval exchange transformation with data (π, λ') where $\lambda' = (\mu(I_i))_{i \in A}$. The map T_μ has exactly the same number of discontinuities as T. Because, T_μ preserves the Lebesgue measure, it is an interval exchange transformation.

Conversely, given $\lambda' \in C(\pi, \lambda)$ we consider the interval exchange transformation $T_{\pi,\lambda'}$. By construction, it also satisfies Keane condition. The orbit of 0 for T_λ and $T_{\lambda'}$ are dense. Moreover the maps $\lambda' \in C(\pi, \lambda) \to T^n_{\pi,\lambda'}(0) - T^m_{\pi,\lambda'}(0)$ are continuous and never 0. Hence The map $T^n_{\pi,\lambda}(0) \to T^n_{\pi,\lambda'}(0)$ is an increasing bijection between dense subsets of $[0, 1]$. It has a unique prolongation to $[0, 1]$. Hence $T_{\pi,\lambda}$ and $T_{\pi,\lambda'}$ are conjugate via an increasing homeomorphism. The pull-back of the Lebesgue measure defines an invariant measure μ so that $(\mu(I_i))_{i \in A} = \lambda'$. □

Corollary 19. *If $T_{\pi,\lambda}$ is a d-interval exchange transformation with no connection then $C(\pi, \lambda)$ has at most $d - 1$ extremal rays. In particular, a d-interval exchange transformation without connection admits at most $d - 1$ ergodic measures.*

Proof. Let $A_n = A_n(\pi, \lambda)$. By definition, each of the cone $A_n\mathbb{R}_+^A$ has exactly d extremal rays. So in the limit there are at most d which are exactly obtained from the projective limits $A_n e_i / \|A_n e_i\|$ where $i \in A$. Recall that $\det(A_n) = 1$ and hence that A_n preserve the volume. Because T has no connection, for all $i \in A$ we have $\|A_n e_i\| \to \infty$. Now, if all limits were different then the volume of $A_n(\{x : x_i \geq 0 \text{ and } x_i \leq 1\})$ would tend to ∞. □

3.3. **Linear recurrence and Boshernitzan condition.** We first introduce two important dynamical notions in the context of symbolic dynamics: linear recurrence and Boshernitzan condition. We then show that Boshernitzan condition implies unique ergodicity.

Definition 20. Let $X \subset A^{\mathbb{N}}$ be a shift with an invariant measure μ. Let $\varepsilon_n(X)$ be the minimum μ-measure of a cylinder of size n, i.e.

$$\varepsilon_n(X) = \min\{\mu([w]) : \; w \in L_X, |w| = n\}.$$

We say that (X, T, μ)

- is **linearly recurrent** (or is of bounded type) if $\inf n\varepsilon_n(X) > 0$,
- satisfies **Boshernitzan condition** if $\limsup n\varepsilon_n(X) > 0$.

Linear recurrence is much stronger than Boshernitzan condition. In the case of interval exchange transformations (endowed with the Lebesgue measure) we have the following alternative definition: $\varepsilon_n(T)$ is the minimum length of the intervals defining T^n.

Given a shift X and a finite word $u \in \mathcal{L}_X$ we denote by $\mathcal{R}_u$ the **return words** to u, that is the set of words w so that $wu \in \mathcal{L}_X$ and wu starts with u. One can think of the return words as the possible coding of orbits when one consider the first return map to the cylinder $[u]$.

Theorem 21 (Boshernitzan [Bos15])**.** *Let (X, T, μ) be a shift. Then the following are equivalent*

(1) (X, T, μ) is linearly recurrent,
(2) there exists a constant C_1 such that for any word u of L_X, $\max\{|w| : w \in \mathcal{R}_u\} \leq C_1|u|$,
(3) there exists a constant C_2 such that any word of length $C_2 n$ of L_X contains all words of length n.

We recall from Fabien Durand's lecture that an important source of linearly recurrent systems are given by substitutive ones. In particular, self-similar interval exchange transformations.

Proposition 22. *A shift generated by a primitive substitution is linearly recurrent. In particular, if an interval exchange transformation $T_{\pi,\lambda}$ has no connection and if its Rauzy induction is periodic, then it is linearly recurrent.*

Proof of Theorem 21. We first prove the equivalence between (2) and (3). Let $u \in L_{X,n}$ and consider $N = n - 1 + \max\{|w| : w \in \mathcal{R}_u\}$. Then any word of length N contains u and there exists a word of length $N - 1$ that does not contain u. Hence we can take $C_2 = C_1 + 1$.

Conversely, if any word of length N contains all word of length n then pick a word that starts with u of length $N + 1$ and remove the first letter. Because of our assumption it contains an other occurrence of u so that $\max\{|w| : w \in \mathcal{R}_u\} \leq N + 1 - n$. Hence $\max\{|w| : w \in \mathcal{R}_u\} \leq (C_2 - 1)n + 1$. So we can take $C_1 = C_2$.

Now, let us prove the equivalence with (1). If the lengths return words are bounded in length by $C_1|u|$ then for any invariant measures μ we have $|u|\mu([u]) \geq 1/C_1$ and hence $\inf n\varepsilon_n \geq 1/C_1$.

Conversely, assume that $\delta = \inf n\varepsilon_n > \varepsilon > 0$. Let u be a word of length n and let $w \in \mathcal{R}_u$ and let $N = |w|$. We want to bound N/n. If $N \leq n$ then there is nothing to prove. So we can assume that $w = us$ with s non-empty. Let us introduce the following set of words of length N for $n \leq k \leq N$

$$W_k = \underbrace{* * \ldots *}_{N-k \text{ times}} w_0 w_1 \ldots w_{k-1}.$$

In other words Y_k is the set of words of length N that ends with the prefix of length k of w By construction, these sets are disjoint and hence $\sum \mu([W_k]) \leq 1$. On the other hand, by invariance of the measure, $\mu([W_k]) = \mu([w_0 w_1 \ldots w_{k-1}]) \geq \frac{\varepsilon}{k}$. So

$$1 \geq \sum_{k=n}^{N} \frac{\varepsilon}{k} \geq \varepsilon \int_{n-1}^{N} \frac{dx}{x} \geq \varepsilon \log\left(\frac{N}{n}\right).$$

Hence $N/n \leq \exp(\varepsilon)$. $\square$

Theorem 23 ([Bos92])**.** *Let (X, T, μ) be a minimal shift that satisfies Boshernitzan condition then it is uniquely ergodic.*

Before proving the Theorem, let us mention that under the hypothesis $\inf n\varepsilon_n > 0$ it is easy to derive unique ergodicity. Indeed, if we had two ergodic measures μ and ν then the ratios $\mu([u])/\nu([u])$ are unbounded. In particular, it contradicts the fact that $\mu([u])$ is controlled by $1/|u|$.

Proof. We proved in Corollary 19 that the number of ergodic measure of interval exchanges are finite. This is also true under the weaker hypothesis that the complexity of the shift is sub-linear (i.e. $\limsup p(n)/n < +\infty$). This more general result is also due to M. Boshernitzan and we refer to the original article [Bos85] (Corollary 1.3) or the chapter [FM10] which is a bit more general.

For two finite words u and v we denote by $|u|_v$ the number of occurrences of v in u.

Let us assume that the shift admits more than one ergodic measure $\{\nu_1, \nu_2, \ldots, \nu_k\}$. There exists a finite word w so that $\nu_1([w]) \neq \nu_2([w])$ and we can assume that $a := \nu_1([w]) < \nu_2([w]) =: b$ and that for all $i = 1, \ldots, k$ we have $\nu_i([w]) \notin (a, b)$.

Let $J = [u, v]$ be a proper subinterval of (a, b), i.e. $a < u < v < b$.

By the Birkhoff theorem, for any $\varepsilon > 0$ we have for any ergodic measure ν that

$$\lim_{n\to\infty} \nu \left\{ x \in X : \frac{|x_{0:n}|_w}{n} - \nu([w]) \right\} = 0 \tag{1}$$

where $x_{0:n} = x_0 x_1 \ldots x_{n-1}$ is the prefix of length n of x. In particular

$$\lim_{n\to\infty} \nu \left\{ x \in X : \frac{|x_{0:n}|_w}{n} \in J \right\} = 0.$$

Since this equality holds for every ergodic measures, it also holds for any invariant measures. In particular,

$$\lim_{n\to\infty} \mu \left\{ x \in X : \frac{|x_{0:n}|_w}{n} \in J \right\} = 0. \tag{2}$$

On the other hand by (1), for n large enough there are two words of length n, U and V in $\mathcal{L}_X$ so that

$$\frac{|U|_w}{n} < u \quad \text{and} \quad \frac{|V|_w}{n} > v. \tag{3}$$

Since X is minimal, there exists a finite sequence of words of lengths n in $\mathcal{L}_{X,n}$ so that $U = W_1, W_2, \ldots, W_m = V$ connecting U with V and such that the suffix of length $n-1$ of W_i coincide with the prefix of length $n-1$ of W_{i+1}. We can assume that there is no repetition in this path.

It is clear that $||W_i|_w - |W_{i+1}|_w| \leq 1$. The last inequality coupled with (3) implies that for at least $[n(v-u)]$ distinct W_i we have $\frac{|W|_w}{n} \in J$. Now, by definition of ε_n we have

$$\mu \left\{ x \in X : \frac{|x_{0:n}|_w}{n} \in J \right\} \geq [n(v-u)]\varepsilon_n.$$

Which contradicts (2) for n large enough. □

Proposition 24. *Let α be irrational. Then the rotation T_α satisfies Boshernitzan condition. Moreover it is linearly recurrent if and only if the continued fraction expansion of α is bounded.*

The proof is given in exercise 19

3.4. **Vorobets identities.** In that section we show identities that relate the behaviours of three quantities of interval exchange transformations T and translation surfaces S:

(1) the quantity $n\varepsilon_n(T)$ that was the main actor of the previous section,
(2) the spread of best approximations of S as the imaginary part goes to infinity,
(3) the systoles of the surface $g_t S$, that is $\operatorname{sys}(g_t S) := \min_{v \in \mathrm{V}(S)} |\operatorname{Re}(v)| + |\operatorname{Im}(v)|$.

The first half of them are due to Vorobets [Vor96] and can also be found in [HMU15].

Theorem 25 (Vorobet's identities)**.** *Let $T_{\pi,\lambda}$ be an interval exchange transformation and $S_{\pi,\lambda,\tau}$ one of its suspensions. Then*

$$\begin{aligned}\frac{1}{|\lambda|} \liminf_{n\to\infty} n\varepsilon_n(T) &= \frac{1}{\operatorname{Area}(S)} \liminf_{v \in \mathrm{BA}(X) : |\operatorname{Im}(v)| \to \infty} |\operatorname{Re}(v)|\, |\operatorname{Im}(v)| \\ &= \frac{1}{\operatorname{Area}(S)} \liminf_{t\to\infty} (\operatorname{sys}(g_t S))^2\end{aligned}$$

and

$$\frac{1}{|\lambda|} \limsup_{n\to\infty} n\varepsilon_n(T) = \frac{1}{\mathrm{Area}(S)} \limsup_{R\to\infty} R \min\{|\operatorname{Re}(v)| : v \in \mathrm{BA}(S), |\operatorname{Im}(v)| \leq R\}$$
$$= \frac{1}{\mathrm{Area}(S)} \limsup_{t\to\infty} (\mathrm{sys}(g_t X))^2$$

The proof of this result is lengthy but rather elementary.

Proof. We first need to introduce a combinatorial analogue of best approximations. Let T be an interval exchange transformation. A *reduced triple* for T is a triple (m, α, β) made of a positive integer m, a singularity α of T and a singularity β of T^{-1} and such that the interval with extremity $T^{-m}(\alpha)$ and β does not contain any of the points $T^{-k}(\alpha')$ where $0 \leq k \leq m$ and α' a singularity of T. A picture is given in Figure 14.

We claim that if n is large enough so that $\varepsilon_n(T) < \min \lambda_i$ then

$$\mathcal{E}_n(T) = \min\{|T^{-m}\alpha - \beta| : \ (m, \alpha, \beta) \text{ is a reduced triple and } m < n\}.$$

Moreover the function $n \mapsto \varepsilon_n(T)$ is decreasing and there is a gap between n and $n+1$ if and only if there is a reduced triple (n, α, β) with $|T^{-n}\alpha - \beta| < \mathcal{E}_n(T)$.

Assuming the claim we obtain easily that if $\mathcal{E}_n(T) < \min \lambda_i$ then

$$\liminf_{n\to\infty} n\varepsilon_n(T) = \liminf\{m|T^{-m}\alpha - \beta| : \ (m, \alpha, \beta) \text{ reduced triple}\}$$

and

$$\limsup_{n\to\infty} n\varepsilon_n(T) = \limsup_{n\to\infty} n \min\{|T^{-m}\alpha - \beta| : \ (m, \alpha, \beta) \text{ reduced triple with } m < n\}.$$

We only sketch the proof of the claim. Let J be an interval of T^n. Then its two extremities are determined by two preimages of singularities of T of $T^{-n_i}\alpha_i^{top}$ and $T^{-n_j}\alpha_j^{top}$. The condition $\mathcal{E}_n(T) < \min \lambda_i$ implies that $n_i \neq n_j$. Without loss of generality, we can assume that $n_i < n_j$. It can then be proven that there exists a reduced triple with length either $n_j - n_i - 1$ or $n_j - n_i - 2$. Conversely, reduced triple forces discontinuities of the power of T. See also Figure 14.

Now we prove that reduced triple are essentially the same thing as best approximations. Let $T = T_{\pi,\lambda}$ be an interval transformation and let $S = S_{\pi,\lambda,\tau}$ be one of its suspensions. Let I be the interval in S on which the Poincaré map is T. To each singularity of T corresponds a vertex of the surface that is reached by following (forward) the linear flow. Similarly, if we follow backward the linear flow from the singularity of T^{-1} we end up in singularities. Moreover, the time needed to bump into the singularity is bounded by $H = \max \sum |\tau_i|$.

To any reduced triple (m, α, β) one can consider the following construction. Let J be the subinterval of I with end points $T^{-m}(\alpha)$ and β. By construction, it is a bottom side of a rectangle with top side J' delimited by α and $T^m(\beta)$. Now we can apply the backward translation flow from the bottom side and the forward one from top. If the first singularities encountered in that process are the one associated to α and β, then we get a best approximation γ.

Conversely, given a best approximation γ one can consider the first and last time it hits I.

We will avoid the proof of the following Lemma.

Lemma 26. *Let $T_{\pi,\lambda}$ be an interval exchange transformation and $S_{\pi,\lambda,\tau}$ a suspension.*

Let n_0 be the first time n so that all intervals of T contains a preimage of a bottom singularity $T^{-k}\alpha$ with $k \leq n$. Then, for any reduced triple (m, α, β) so that $m \geq n_0$ is associated a best approximation as described in the above procedure. Conversely, any best approximation γ whose slope $|\operatorname{Im}(\gamma)|/|\operatorname{Re}(\gamma)|$ is larger than the slope of any of the τ_i for i in A is associated to a unique reduced triple.

Moreover, if γ denote the best approximation associated to some reduced triple (m, α, β) one has

$$\operatorname{Re}(\gamma) = T^{-m}(\alpha) - \beta \quad \text{and} \quad Im(\gamma) = S_m(h, x) + h_{start} + h_{end}$$

where $S_m(h, x)$ is the Birkhoff sum of the height function of the suspension construction and x is any point in the interval determined by $T^{-m}\alpha$ and β and h_{start} is the height of the vertex below β and h_{end} is the height of the vertex above α.

Lemma 26 shows that the spread of a best approximation is equal up to some additive constant is the product $S_m(\tau, x)|T^{-m}(\alpha) - \beta|$. In order to prove that it is asymptotically equal to $m|T^{-m}(\alpha) - \beta|$ one

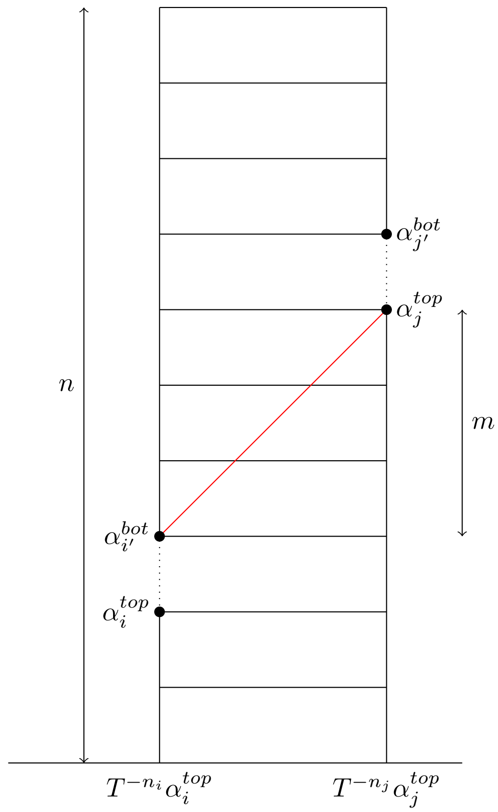


FIGURE 14. $n\varepsilon_n$ and reduced triples.

actually needs to involve unique ergodicity. As it holds in the case of Boshernitzan condition (Theorem 23) then any behaviour of $m|T^{-m}(\alpha) - \beta|$ is the same as the behaviour of the spread of best approximations.

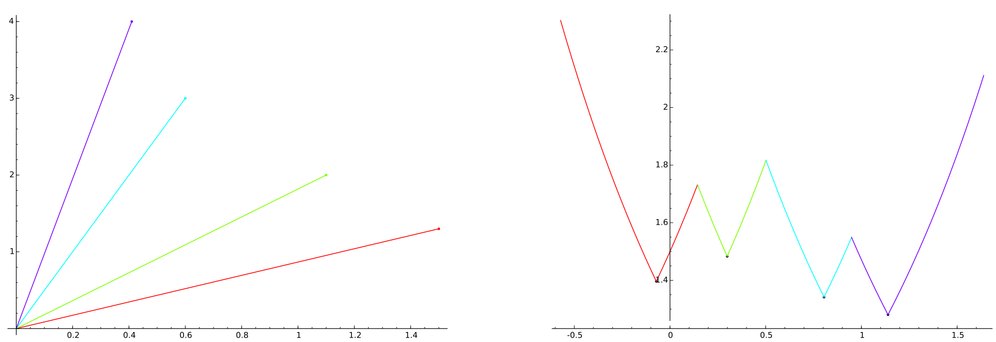


FIGURE 15. Local extrema of the L^∞-systole illustrated with four vectors. The left picture represents four vectors $V = \{v_1, v_2, v_3, v_4\}$ in $\mathbb{R}^2$ and the right picture is $t \mapsto \mathrm{sys}(g_t V)$. Each color in the right picture corresponds to times during which a given vector on the left is the shortest one. The local minima are when the vector is diagonal and the local maxima when there are simultaneously two vectors of the same length.

Now we show how to relate best approximations with systoles. Given the L^∞ norm on $\mathbb{R}^2$, and a vector $v = (x, y)$ with $x > 0$ and $y > 0$ the function $v \mapsto \|g_t v\|$ can be explicitly written as

$$\|g_t v\|_\infty = \begin{cases} e^t x & \text{if } t \geq (\log(y) - \log(x))/2, \\ e^{-t} y & \text{if } t \leq (\log(y) - \log(x))/2. \end{cases}$$

It is easily seen that

- the local minima of the function $t \mapsto \mathrm{sys}(g_t S)$ are obtained when there is a best approximation so that $|\operatorname{Re}(g_t\gamma)| = |\operatorname{Im}(g_t\gamma)|$. In that case we have $\mathrm{sys}(g_t S)^2 = |\operatorname{Re}(g_t\gamma)||\operatorname{Im}(g_t\gamma)| = |\operatorname{Re}(\gamma)||\operatorname{Im}(\gamma)|$.
- The local maxima of the function $t \mapsto \mathrm{sys}(g_t S)$ are obtained when there are two consecutive best approximations γ and γ' so that $|\operatorname{Re}(g_t\gamma)| = |\operatorname{Im}(g_t\gamma')|$. In that case we have $\mathrm{sys}(g_t S)^2 = |\operatorname{Re}(g_t\gamma)||\operatorname{Im}(g_t\gamma')| = R\min\{|\operatorname{Re}(\gamma'')| : \gamma'' \in \mathrm{BA}(S) \text{ with } \operatorname{Im}(\gamma'') < \operatorname{Im}(\gamma')\}$.

One can have a look at Figure 15. □

3.5. **Notes and further results.** The first examples of non-uniquely ergodic interval exchange transformations are due to M. Keane [Kea77]. Then, in the context of billiards Y. Cheung and H. Masur [CM06] exhibited many other examples (that are intimately related to the Veech skew products of [Vee68]). The sharp number for the number of invariant measures was obtained by Katok [Kat73] and is actually given by the genus of the surface. The fact that it is sharp in any stratum is due to J. Fickenscher [Fic14].

In the mood of Boshernitzan criterion, there is a little bit finer result due to Masur (see [MT02]) but that has up to now no symbolic counterpart. There are also finer results that shows that if $n\varepsilon_n$ goes slowly to 0 then the interval exchange transformation is uniquely ergodic ([CE07], [Tre14]). Again, this has no symbolic counterpart.

The linear recurrence property can also be expressed as a property of the matrices appearing in the Rauzy induction (see [HMU15], [KM14]). In a symbolic context, see Theorem 6.5.10 in [Dur10].

Up to the speaker knowledge, it is not known whether the Boshernitzan condition has a nice formulation in terms of the matrices of the Rauzy induction.

4. Some generic properties of interval exchange transformations

Given a dynamical property (P) (like linear recurrence, unique ergodicity), we can consider the two following questions

(1) what can be said about the set of translation surfaces in a given stratum for which their (vertical) translation flow satisfies (P)?
(2) given a translation surface, what can be said about the set of directions θ for which the translation flow in direction θ satisfies (P)?

The results of the second kind are much stronger. They are also much more suited to any concrete problems involving rational billiards.

4.1. **How do we prove something for a generic translation surface?** To prove results of the first kind, the strategy makes use of the invariant measure on strata (the Masur-Veech measure) and the ergodicity of the $\mathrm{SL}(2,\mathbb{R})$-action (Masur-Veech theorem).

It has first been applied to prove the Keane conjecture

Theorem 27 (Keane's conjecture, Masur-Veech theorem [Mas82],[Vee82])**.** *If $\pi \in S_r$ is irreducible, then for almost every $\lambda \in \Delta_r$ the interval exchange transformation $T_{\pi,\lambda}$ is uniquely ergodic.*

We will see in the next section a strengthen version of this result.

4.2. **Some results that hold for *all* translation surfaces.** In this section we mention three important results that hold for *all* translation surfaces.

Theorem 28 (Masur asymptotic)**.** *Let S be a translation surface. Then there exists constants c_1 and c_2 so that for any $R > 0$*

$$c_1 R^2 \leq \#V(S,R) \leq c_2 R^2.$$

The following result is an enhancement of the Masur-Veech theorem (Theorem 27).

Theorem 29 (Kerckhoff-Masur-Smillie theorem, [KMS86])**.** *Let S be a translation surface. Then for almost every θ the flow in direction θ in S is uniquely ergodic.*

Theorem 30 ([KW04], [CCM13])**.** *Let S be a translation surface. Then the set Λ of direction θ for which $r_\theta S$ is linearly recurrent (i.e. the geodesic $\{g_t r_\theta S\}_{t\geq 0}$ is bounded) is thick: for any open set $U \subset S^1$, the Hausdorff dimension of $U \cap \Lambda$ has Hausdorff dimension 1.*

4.3. **Masur asymptotic theorem for tori.** In this section we give a proof of a very special case of Masur asymptotic theorem (Theorem 28).

Theorem 31. *Let S be a torus of area one. Then*

$$\# V(S,R) \sim \frac{3}{\pi} R^2.$$

Note that in this particular case we get an exact asymptotic. This precise asymptotic is known to holds for the so-called Veech surfaces.

Proof. Let Λ be the set of primitive vectors in $\mathbb{Z}^2$. By definition $\mathbb{Z}^2$ is the disjoint union of $\{0\}$, Λ, 2Λ, etc.

Now let Ω be a compact set in $\mathbb{R}^2$ which contains 0 in its interior. For a discrete set $\Gamma \subset \mathbb{R}^2$ let

$$\delta^-(\Gamma) := \liminf_{R\to\infty} \frac{\#(R\Omega \cap \Gamma)}{R^2} \quad \text{and} \quad \delta^+(\Gamma) := \limsup_{R\to\infty} \frac{\#(R\Omega\cap\Gamma)}{R^2}$$

It is easy to see that

(1) $\delta^-(\Gamma) \leq \delta^+(\Gamma)$,
(2) $\delta^-(\Gamma_1 \cup \Gamma_2) \geq \delta^-(\Gamma_1) + \delta^-(\Gamma_2)$,
(3) $\delta^+(\Gamma_1 \cup \Gamma_2) \leq \delta^+(\Gamma_1) + \delta^+(\Gamma_2)$,
(4) $\delta^\pm(n\Gamma) = 1/n^2 \delta^\pm(\Gamma)$.

$$\liminf_{R\to\infty} \#(R\Omega \cap n\Lambda) = \frac{1}{n^2} \liminf_{R\to\infty} \#(R\Omega \cap \Lambda)$$

As $\delta^-(\mathbb{Z}^2) = \delta^+(\mathbb{Z}^2) = \mathrm{Area}(\Omega)$, applying these inequality to $\mathbb{Z}^2 = \{0\} \cup \Lambda \cup (2\Lambda) \cup \ldots$ we got

$$\delta^+(\Lambda) \sum_{n\geq 1} \frac{1}{n^2} \leq \mathrm{Area}(\Omega) \geq \delta^-(\Lambda) \sum_{n \geq 1} \frac{1}{n^2}.$$

Now since $\delta^-(\Lambda) \leq \delta^+(\Lambda)$ we obtain that

$$\delta^-(\Lambda) = \delta^+(\Lambda) = \mathrm{Area}(\Omega) \frac{6}{\pi^2}.$$

To prove the theorem, just apply this with Ω being the unit ball. □

4.4. **Sketch of a proof of Kerckhoff-Masur-Smillie theorem.** The original proof can be found in [Mas90, Mas88]. A finer result can be found in [EM01] and a quantitative statement in [Vor97]. The original proof of Kerckhoff-Masur-Smillie did not use this technology. Though, their strategy is very close in spirit.

Proof of Theorem 29. We show that in almost every direction θ the linear flow ϕ_t^θ satisfies Boshernitzan condition.

Let us introduce for $t > 0$ and $0 < \varepsilon < 1/2$ the following set

$$\mathrm{BAD}(t) = \left\{ \theta \in [0, 2\pi] : \limsup_{t\to\infty} \mathrm{sys}(g_t r_\theta S) < \varepsilon \right\}.$$

We claim that there exists a constant C so that

$$\mathrm{Leb}(\mathrm{BAD}(t)) = C\varepsilon^2.$$

Let us first explain why the claim implies the result. Assuming the claim, we obtain that

$$\mathrm{Leb}\{\theta \in [0,2\pi] : \limsup_{t\to\infty} \mathrm{sys}(g_t r_\theta S) \geq \varepsilon\} \geq 1 - C\varepsilon^2.$$

As ε was arbitrary, we got that Lebesgue-almost every θ satisfies the Boshernitzan condition.

We now prove the claim. From Theorem 28: there exists a constant c_3 so that

$$\sum_{v \in V(S,R)} \frac{1}{\|v\|} \leq c_3 R.$$

We can also assume that $\min\{\|v\| : v \in V(S,R)\} \geq 1$.

Given an element $v \in V(S,R)$, the measure of the set of angles θ such that $\|g_t r_\theta v\| < \varepsilon$ clearly only depends on $\|v\|$. Let us do the computation for $v = (0, |v|)$. A simple computation gives that

$$g_t r_\theta v = \|v\| \begin{pmatrix} -\sin(\theta) e^t \\ \cos(\theta) e^{-t} \end{pmatrix}.$$

So $\|g_t r_\theta v\| < \varepsilon$ implies that $\|v\| |\sin(\theta)| e^t < \varepsilon$ and $\|v\| |\cos(\theta)| e^{-t} < \varepsilon$.

As $\|v\| \geq 1$, we have $|\sin(\theta)| < \varepsilon < 1/2$ and hence $\cos(\theta) \geq \sqrt{2}/2 > 1/2$. The second inequality hence gives $\|v\| < 2\varepsilon e^t$. Now, using the first inequality we obtain that $|\theta| < 2|\sin(\theta)| < \varepsilon e^{-t}/\|v\|$.

Hence,

$$\mathrm{Leb}(BAD(t)) \leq \sum_{v \in V(S, 2\varepsilon e^t)} \frac{4\varepsilon e^{-t}}{\|v\|} \leq 8 c_3 \varepsilon^2.$$

□

4.5. **Notes and further results.** Much more is known about the growth rate of $|V(S,R)|$. In [EM01] it is proven that in a given component of stratum $\mathcal{C}$ there exists a constant $c_{SV}(\mathcal{C})$ (the Siegel-Veech constant) so that for almost every translation surface S in that component we have $|V(S,R)| \sim c_{SV}(\mathcal{C}) R^2$. The asymptotic is also known to hold (with a different constant) for the so called *Veech surfaces*, see [Vee89]. It is currently unknown whether there is an exact quadratic growth for all surfaces!

Recently, a very deep theorem has been proved which allowed to use ergodic methods to prove results about *all* surfaces. Eskin and Mirzakhani showed that the $\mathrm{SL}(2,\mathbb{R})$-orbit of *any* translation surface somehow equidistributes to some "nice" measure on the stratum. The question about individual surfaces can hence be studied by mean of the $\mathrm{SL}(2,\mathbb{R})$-invariat measure on strata. The Eskin-Mirzakhani is deep and long and is one of the important result that leads to the attribution of the Fields medal to Maryam Mirzakhani in 2015.

5. Further reading and some open questions

We avoided many important topics of interval exchange transformations. Here is a short list of suggestions for further reading that were not already mentioned. Note that they are often harder to read than the already mentioned results.

- [AF07]: weak-mixing of interval exchanges and translation flows;
- [Buf14]: for limit laws of Birkhoff sums (see also [DHL14]);
- [EC15]: an important application of Eskin-Mirzhakani-Mohammadi results that show that the Birkhoff theorems for the Teichmüller flow actually holds in any Teichmüller discs;
- [AC12]: for a study of the asymptotic distribution of the gap between angles of vectors in $V(S,R)$.

6. Exercises

Most of the exercise are simple. We use the convention of adding a (*) before a more difficult question, (**) before a question which needs a certain amount of work and (***) for the ones for which there is no known answer up to the author knowledge.

6.1. Word combinatorics and coding of interval exchange transformations.

Exercise 1 (recurrence, transitivity, connectedness)
Let $L \subset A^*$ be a language.

(1) Prove that the following are equivalent
  (a) the associated shift X_L is transitive (i.e. it admits a dense orbit),
  (b) for all words $u, v \in L$ there exists w so that $uwv \in L$,
  (c) there exists a recurrent word $u \in A^{\mathbb{N}}$ so that $L = L_u$.
(2) Under the above condition, prove that all Rauzy graphs are strongly connected
(3) Find a counterexample to the converse of (2).

Exercise 2 (uniform recurrence and minimality)
In this exercise we see that uniform recurrence and minimality are actually equivalent.

(1) Let $L \subset A^*$ be a language. Prove that the associated subshift X_L is minimal (i.e. all orbits are dense) if and only if L is uniformly recurrent.
(2) Prove that the natural coding of a rotation T (or more generally of an interval exchange transformation) is uniformly recurrent, if and only if all infinite orbits of T are dense.

Exercise 3
Let (X,T) be a topological dynamical system with X compact metric. Prove that the following are equivalent.

(1) for any point $x \in X$ its forward orbit $\{x, Tx, T^2x, \ldots\}$ is dense in X,
(2) for any non-empty open set U we have $\bigcup_{k=0}^{\infty} T^{-k} U = X$,

(3) for any non-empty open set U there exists n so that $\bigcup_{k=0}^{n} T^{-k}U = X$.

Exercise 4 (Morse-Hedlund theorem)
Let $u \in A^{\mathbb{N}}$ be an infinite word on some finite alphabet A. Prove that the following are equivalent

(1) u is ultimately periodic, in other words $u = pvvv\ldots$ for some finite words p and v.
(2) $p_u(n)$ is bounded,
(3) for some n we have $p_u(n) \leq n$.

Exercise 5
Let $T_\alpha : [0,1] \to [0,1]$ be the rotation by α. Let $\mathcal{L}_\alpha$ be the language of the natural coding on $\{A, B\}$.

(1) Show that $\mathcal{L}_\alpha$ is closed under the reversal operation: $u_0 u_1 \ldots u_{n-1} \mapsto u_{n-1} \ldots u_1 u_0$.
(2) Show that for any length n there are exactly one word u in $\mathcal{L}_\alpha$ so that both of ua and ub belongs to $\mathcal{L}_\alpha$.
(3) Show that there exists a unique right infinite word u^+ and a unique left infinite word u^- so that all of Au^+, Bu^+, u^-A and u^-B are coding of an orbit of T^α.
(4) Show that there both u^-ABu^+ and u^-BAu^+ code (singular) orbits of T_α.
(5) Show that Au^+ (respectively Bu^+) is the minimal (respectively maximal) word for the lexicographic ordering in X_α.

6.2. **Permutations and Rauzy diagrams.**

Exercise 6
We denote by S_d the permutations of $\{1, 2, \ldots, d\}$. We recall that $\#S_d = d! = d \cdot (d-1) \cdot \ldots \cdot 1$ the factorial number. We consider the permutations as the array of numbers $(\pi(1)\pi(2)\ldots\pi(d))$.

Let us say that a permutation $\pi \in S_d$ is irreducible if there is no $k < d$ so that $\pi(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$.

We introduce the following concatenation on permutations $\pi \in S_d$ and $\pi' \in S_{d'}$

$$\pi \cdot \pi' = (\pi(1)\pi(2)\ldots\pi(d)\pi'(1) + d\pi'(2) + d\ldots\pi'(n') + d).$$

(1) Show that $\pi \cdot \pi' \in S_{d+d'}$.
(2) Show that a permutation is irreducible if and only if it can not be written as the concatenation of two other permutations.
(3) Make the list of irreducible permutations for $d = 2, 3, 4$.
(4) Show that any permutation in S_d can be uniquely factorized into a product of irreducible permutation.
(5) Using the previous item, construct a recursive formula to compute the number of irreducible permutations.
(6) What is the number of irreducible permutations in S_5? in S_6?
(7) (**) What is the asymptotic behaviour of the number of irreducible permutations?

Exercise 7 (Rauzy diagrams)
In this exercise we make the list of small Rauzy diagrams.

(1) If $\pi = (\pi^{top}, \pi^{bot})$ is the combinatorial data of a d-interval exchange transformation, show that $\pi^{bot} \circ (\pi^{top})^{-1}$ is a permutation in S_d.
(2) Show that $\pi = (\pi^{top}, \pi^{bot})$ is irreducible if and only if $\pi^{bot} \circ (\pi^{top})^{-1}$ is irreducible with the definition given in Exercise 6.
(3) With the help of Exercise 6, make the list of Rauzy diagrams for $d = 2, 3, 4$ (you can consider permutations (π^{top}, π^{bot}) up to relabelling, that way you will end up with smaller graphs).

Exercise 8
We consider the permutation $\pi = (ABC/CBA)$.

(1) Draw the Rauzy diagram of π (it has 3 vertices) and compute the substitutions associated to the Rauzy induction,
(2) Consider the path of induction of length 6 $\gamma = (\pi, tbtbtb)$ (where we use t for top and b for bottom). Compute the associated substitution and matrix.
(3) Compute the length of the interval exchange transformation associated to γ.
(4) Compute the substitution associated to γ and write down the first words of the language.
(5) Identify the two infinite left and right special words.

6.3. **Dynamics.**

Exercise 9

Let X be a compact metric space and $T : X \to X$ a continuous map that preserves a measure μ.

(1) Show that if T admits a closed invariant set Y then it admits an invariant measure supported on Y.
(2) Assume that T admits an ergodic invariant measure μ that gives positive mass to open sets. Show that for μ-almost every point $x \in X$ its orbit is dense.

Exercise 10

This exercise is a counterpart of Exercise 9.

(3) Construct an example of a uniquely ergodic system which is not minimal.
(4) (*) Construct an example of a minimal system which is not uniquely ergodic.

Exercise 11

Let X be a compact metric space and $T : X \to X$ a continuous map that preserves a measure μ. Now we assume that the measure μ gives positive map to open set. Show that Poincaré recurrence theorem implies that for μ-almost every point x, we have

$$\liminf_{n\to\infty} \operatorname{dist}(T^n x, x) = 0.$$

Exercise 12 (Kac s lemma)

Let (X, T, μ) be a measurable dynamical system. Let $Y \subset X$ be a subset of positive measure. Let $r_Y(x) = \min\{n \geq 0 :\ T^n x \in Y\}$ and $r_Y^+ = \min\{n > 0 :\ T^n x \in Y\}$.

(1) Prove that $r_Y < \infty$ and $r_Y^+ < \infty$ almost everywhere.
(2) Using the fact that the sets $r_Y^{-1}(\{n\})$ are disjoint, prove that

$$\int_Y r_Y^+(x) d\mu(x) \leq 1.$$

(3) Assuming that μ is ergodic, show that the above inequality is actually an equality.

6.4. **Rotations.**

Exercise 13

Prove Lemma 3 about canonical basis of tori.

Exercise 14

Let ζ_A and ζ_B be the vectors of a suspension of a rotation. Show that the set of best approximations of this torus is exactly the set $\{\zeta_A^{(n)} - \zeta_B^{(n)}\}_{n\in\mathbb{Z}}$ where $\zeta^{(n)}$ is the sequence of suspension vectors of the Rauzy-Veech induction starting from $\zeta_0 = \zeta$.

Exercise 15

Let $T : [0,1] \to [0,1]$ defined by $Tx = \left\{\frac{1}{x}\right\}$ be the Gauss map and let $a(x) = \left\lfloor \frac{1}{x} \right\rfloor$. In other words we have $Tx = \frac{1}{x} - a(x)$.

We set $a_0(x) = 0$, $a_1(x) = a(x)$ and $a_n(x) = a(T^{n-1}x)$. From these **partial quotients** of x we define inductively a sequence of rational numbers p_n, q_n by setting $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$ and

$$p_n = a_n p_{n-1} + p_{n-2} \quad \text{and} \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Given a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{GL}(2,\mathbb{Z})$ we associate the **homography** $f_A : x \mapsto \frac{ax+b}{cx+d}$. This function is well defined in $\mathbb{R} \cup \{\infty\}$.

(1) Show that composition of homography correspond to matrix multiplication in other words $f_A \circ f_B = f_{AB}$.
(2) Remark that $Tx = f_{A(x)^{-1}}(x)$ where $A(x) = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$.
(3) Show that

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}$$

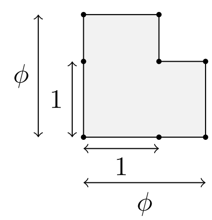
(4) Deduce that $T^n x = (-1)^{n+1} \frac{q_n x - p_n}{q_{n-1}x - p_{n-1}}$.

(5) Show that
$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}}}$$
(6) Using the matrix product expression, deduce that if $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$ then $q_{n-1}/q_n = [0; a_n, a_{n-1}, \ldots, a_1]$.

6.5. **Linear recurrence and Boshernitzan condition.**

Exercise (SageMath) 16 (1) Program a function `def insert(l,x)` that given a sorted list `l` of floating point numbers and a floating point element `x` insert the element `x` in the list in order to keep it sorted and return the position at which it was inserted (hint: use a dichotomy search).
(2) Let $X_1, X_2, \ldots$ be uniform independent random variables in $[0,1]$. The elements $\{0, 1, X_1, X_2, \ldots, X_{n-1}\}$ cut $[0,1]$ into n subintervals. Let ε_n denote the length of the smallest interval. Plot the sequence $n\varepsilon_n$ for some realizations of this process (hint: the function `random` generate a pseudo-random number uniformly in $[0,1]$).
(3) Now pick n fixed and large, plot the experimental distribution of the length of the subintervals (hint: use the command `histogram`).

Exercise 17
Let S be the following L-shaped translation surface where opposite sides are glued together.



Draw the holonomies of saddle connections of length less than 10.

Exercise (SageMath) 18
In this exercise we experiment the values of $n\varepsilon_n$ for rotations. See also Exercise 19 for a more theoretical exercise.

(1) Using the function `def insert(l,x)` from Exercise 16 program a function that given a random number α compute the first terms of the sequence $n\varepsilon_n(\alpha)$.
(2) Plot the sequence as a function of n and identifies the position of the local extrema.
(3) Verify the formulas given in Exercise 19.
(4) Do another function that also works for interval exchange transformations.

Exercise 19
This exercise is a continuation of Exercise 15. See also Exercise 18 for a more experimental exercise.

(1) (*) Show that
$$\varepsilon_n(R_\alpha) = \{q_k \alpha\}$$
where k is the smallest integer so that $q_k \geq n$.
(2) Deduce the following equalities
$$\liminf_{n\to\infty} n\varepsilon_n(T_\alpha) = \liminf_{n\to\infty} q_n\{q_n\alpha\} \quad \text{and} \quad \limsup_{n\to\infty} n\varepsilon_n(T_\alpha) = \limsup_{n\to\infty} q_{n+1}\{q_n\alpha\}$$
(hint: the quantity $\varepsilon_n(R_\alpha)$ changes exactly at the values q_n where it passes from $\{q_{n-1}\alpha\}$ to $\{q_n\alpha\}$).
(3) Prove the following equalities
$$q_n\{q_n\alpha\} = \frac{1}{[a_{n+1}; a_{n+2}, a_{n+3}, \ldots] + [0; a_n, a_{n-1}, \ldots]}$$

and

$$q_{n+1}\{q_n\alpha\} = \cfrac{1}{1 + \cfrac{1}{[a_{n+2}; a_{n+3}, a_{n+4}, \ldots] \times [a_{n+1}; a_n, a_{n-1}, \ldots]}}$$

(4) Prove that

$$\liminf_{n\to\infty} n\varepsilon_n(T_\alpha) \in \left[0, \frac{1}{\sqrt{5}}\right] \qquad \text{and} \qquad \limsup_{n\to\infty} n\varepsilon_n(T_\alpha) \in \left[\frac{5+\sqrt{5}}{10}, +\infty\right].$$

(5) Prove that $\liminf n\varepsilon_n(\alpha) = 0$ if and only if $\limsup n\varepsilon_n(\alpha) = \infty$ if and only if the continued fraction of α is unbounded.

(6) Prove that for $\alpha = (3-\sqrt{5})/2$ we have $\liminf n\varepsilon_n(\alpha) = \frac{1}{\sqrt{5}} \simeq 0.447$ and $\limsup n\varepsilon_n(\alpha) = \frac{5+\sqrt{5}}{10} \simeq 0.7236$.

(7) Deduce Proposition 24. That is, the rotation T_α is linearly recurrent if and only if the partial quotient of α are bounded.

NOTE: The sets $\{\liminf n\varepsilon_n(\alpha); \alpha \in [0,1]\}$ and $\{\limsup n\varepsilon_n(\alpha); \alpha \in [0,1]\}$ are respectively called the *Lagrange* and *Dirichlet* spectrum. They have a complicated fractal like structure.

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