COBHAM'S THEOREM AND SUBSTITUTION SUBSHIFTS

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Abstract. This lecture intends to propose a first contact with subshift dynamical systems through the study of a well known family: the substitution subshifts. This will include an short introduction to topological dynamical systems and combinatorics on words. We will focus on the unique ergodicity of substitution subshifts and we will obtain, as a corollary, a proof of a seminal result on automata theory: the Cobham's theorem.

1. Introduction

The seminal theorem of Cobham has given rise during the last 40 years to a lot of works about non-standard numeration systems and has been extended to many contexts. The original Cobham's theorem is concerned with integer base numeration systems. In this paper, as a result of fifteen years of improvements, we obtain a complete and general version for the so-called substitutive sequences.

A set $E \subset \mathbb{N}$ is *p-recognizable* for some $p \in \mathbb{N} \setminus \{0, 1\}$, if the language consisting of the p-ary expansions of the elements in E is recognizable by a finite automaton. It is obvious to see that E is recognizable if and only if it is p^k-recognizable. In 1969, A. Cobham obtained the following remarkable theorem.

Cobham's theorem. [Cobham 1969] *Let $p, q \geq 2$ be two multiplicatively independent integers (i.e., $p^k \neq q^\ell$ for all integers $k, \ell > 0$). A set $E \subset \mathbb{N}$ is both p-recognizable and q-recognizable if and only if E is a finite union of arithmetic progressions.*

In [Cobham 1972], Cobham made precise the structure of these p-recognizable sets: they are exactly the images by letter-to-letter morphisms of constant-length p substitution fixed points. He also defined the notion of p-automatic sequences: The n-th term of the sequence is a mapping of the last reached state of the automaton when its input is the digits of n is some given base p numeration system. Clearly $E \subset \mathbb{N}$ is p-recognizable if and only if its characteristic sequence is p-automatic. Automata provide a nice and easy description of p-recognizable sets whereas substitutions afford an algorithm to produce such sets.

Thanks to this characterisation, Cobham's theorem can be reformulated in an equivalent in terms of substitution.

Cobham's theorem (Substitutive version). [Cobham 1969] *Let $p, q \geq 2$ be two multiplicatively independent integers and A be a finite alphabet. A sequence $x \in A^{\mathbb{N}}$ is both p-automatic and q-automatic if and only if $x = uvvv\ldots$ for some words u and$v$.*

It is interesting to recall what S. Eilenberg wrote in his book [Eilenberg 1974]: *The proof is correct, long and hard. It is a challenge to find a more reasonable proof of this fine theorem.* Many other proofs have been proposed in this direction. We refer to the the dedicated chapter in [Allouche and Shallit 2003] for an expository presentation.

The goal of this lecture is to present a (simple) proof of Cobham's theorem under some additional hypothesis using dynamical systems and more precisely: subshifts. We will prove the following results.

Theorem 1. *Let σ and τ be two primitive substitutions with dominant eigenvalues α and β respectively. Suppose α and β are multiplicatively independent. Then, (X_σ, T) and (X_τ, T) have a common factor if, and only if, this factor is periodic.*

Corollary 2. *Let σ and τ be two primitive substitutions with dominant eigenvalues α and β respectively. Suppose α and β are multiplicatively independent. Then, (X_σ, T) and (X_τ, T) are isomorphic if, and only if, they X_σ and X_τ are finite with the same cardinality.*

We will left as an exercise (not difficult but not so easy) to prove that Cobham's theorem is a corollary of Theorem 1.

2. Words, substitutions and dynamical systems

2.1. **Words, sequences and morphisms.** We call **alphabet** a finite set of elements called **letters**. Let A be an alphabet, a **word** on A is an element of the free monoïd generated by A, denoted by A^*, i.e. a finite sequence (possibly empty) of letters. Let $x = x_0x_1 \cdots x_{n-1}$ be a word, its **length** is n and is denoted by $|x|$. The **empty word** is denoted by ϵ, $|\epsilon| = 0$. The set of non-empty words on A is denoted by A^+. If $J = [i, j]$ is an interval of $\mathbb{N} = \{0, 1 \cdots\}$ then x_J denotes the word $x_i x_{i+1} \cdots x_j$ and is called a **factor** of x. Analogous definitions hold for open or semi-open intervals. We say that x_J is a prefix of x when $i = 0$ and a suffix when $j = n-1$. If u is a factor of x, we call **occurrence** of u in x every integer i such that $x_{[i,i+|u|-1]} = u$. Let u and v be two words, we denote by $L_u(v)$ the number of occurrences of u in v.

The elements of $A^{\mathbb{N}}$ are called **sequences**. For a sequence $\mathsf{x} = (\mathsf{x}_n; n \in \mathbb{N}) = \mathsf{x}_0\mathsf{x}_1 \cdots$ we use the notation x_J and the terms "occurrence" and "factor" exactly as for a word. The set of factors of length n of x is written $L_n(\mathsf{x})$, and the set of factors of x, or **language** of x, is represented by $L(\mathsf{x})$; $L(\mathsf{x}) = \cup_{n \in \mathbb{N}} L_n(\mathsf{x})$. The sequence x is **periodic** if it is the infinite concatenation of a word v: $\mathsf{x} = vvv \cdots$. It is **ultimately periodic** is $\mathsf{x} = uvvv \cdots$. A **gap** of a factor u of x is an integer g which is the difference between two successive occurrences of u in x. We say that x is **uniformly recurrent** if each factor has bounded gaps.

Let A, B and C be three alphabets. A **morphism** τ is a map from A to B^*. Such a map induces by concatenation a map from A^* to B^*. If $\tau(A)$ is included in B^+, it induces a map from $A^{\mathbb{N}}$ to $B^{\mathbb{N}}$. All these maps are written τ also.

To a morphism τ, from A to B^*, is associated the **incidence matrix** $M_\tau = (m_{i,j})_{i \in B, j \in A}$ where $m_{i,j}$ is the number of occurrences of i in the word $\tau(j)$. To the composition of morphisms corresponds the multiplication of matrices. For example, let $\tau_1 : B^* \to C^*$, $\tau_2 : A^* \to B^*$ and $\tau_3 : A^* \to C^*$ be three morphisms such that $\tau_1 \circ \tau_2 = \tau_3$ (we will quite often forget the composition sign), then we have the following equality: $M_{\tau_1} M_{\tau_2} = M_{\tau_3}$. In particular, if τ is a morphism from A to A^* we have $M_{\tau^n} = M_\tau^n$ for all non-negative integers n.

2.2. **Substitutions.** A **substitution** on the alphabet A is an endomorphism $\sigma : A^* \to A^*$ satisfying:

(1) There exists $a \in A$ such that a is the first letter of $\sigma(a)$;
(2) For all $b \in A$, $\lim_{n\to+\infty} |\sigma^n(b)| = +\infty$.

Note that Condition (1) is not always required in the literature about substitutions. We prefer to use this definition in order to avoid details that are not necessary for the purpose of this lecture. The substitution σ can be extended by concatenation to a map from $A^{\mathbb{N}}$ to $A^{\mathbb{N}}$ we continue to denote σ.

The **language** of σ is the set $L(\sigma)$ consisting of all the words having an occurrence in some $\sigma^n(b)$, $n \in \mathbb{N}$ and $b \in A$.

In some papers (see [Pansiot 1986] for example) the condition (2) is not required to be a substitution and our definition corresponds to what Pansiot call **growing substitutions** in [Pansiot 1986].

We say a square matrix $M = (m_{ij})$ is **primitive** if there exists some n such that M^n has strictly positive entries. Whenever the matrix associated to τ is primitive we say that τ is a **primitive substitution**. It is equivalent to the fact that there exists n such that for all a and b in A, a has an occurrence in $\sigma^n(b)$. Note that in this case $L(\sigma) = L(\mathsf{x})$ for all fixed points x of σ.

Let B be another alphabet, we say that a morphism $\phi : A^* \to B^*$ is a **letter to letter morphism** when $\phi(A)$ is a subset of B. Then the sequence $\phi(\mathsf{x})$ is called **substitutive**, and **primitive substitutive** if τ is primitive.

2.3. **Dynamical systems and subshifts.** By a **dynamical system** we mean a pair (X, S) where X is a compact metric space and S a continuous map from X to itself. We say that it is a **Cantor system** if X is a Cantor space. That is, X has a countable basis of its topology which consists of closed and open sets and does not have isolated points. The system (X, S) is **minimal** whenever X and the empty set are the only S-invariant closed subsets of X. We say that a minimal system (X, S) is **periodic** whenever X is finite. We say it is p-periodic if $\#(X) = p$.

Let (X, S) and (Y, T) be two dynamical systems. We say that (Y, T) is a **factor** of (X, S) if there is a continuous and onto map $\phi : X \to Y$ such that $\phi \circ S = T \circ \phi$ (ϕ is called **factor map**). If ϕ is one-to-one we say that ϕ is an **isomorphism** and that (X, S) and (Y, T) are **isomorphic**.

In this paper we deal with Cantor systems called **subshifts**. Let A be an alphabet. We endow $A^{\mathbb{N}}$ with the infinite product of the discrete topologies. It is a metrizable topology, a metric being given by

$$d(\mathsf{x}, \mathsf{y}) = \frac{1}{2^n} \quad \text{with} \quad n = \inf\{|k|; \mathsf{x}_k \neq \mathsf{y}_k\},$$

where $\mathsf{x} = (\mathsf{x}_n; n \in \mathbb{N})$ and $\mathsf{y} = (\mathsf{y}_n; n \in \mathbb{N})$ are two elements of $A^{\mathbb{N}}$. By a **subshift** on A we shall mean a pair $(X, T_{|X})$ where X is a closed T-invariant ($T(X) = X$) subset of $A^{\mathbb{N}}$ and T is the **shift transformation**

$$\begin{array}{rccl} T & : & A^{\mathbb{N}} & \to & A^{\mathbb{N}} \\ & & (\mathsf{x}_n; n \in \mathbb{N}) & \mapsto & (\mathsf{x}_{n+1}; n \in \mathbb{N}). \end{array}$$

We call language of X the set $L(X) = \{\mathsf{x}_{[i,j]}; \mathsf{x} \in X, i \leq j\}$. Let u be a word of A^*. The set

$$[u]_X = \{\mathsf{x} \in X; \mathsf{x}_{[0,|u|-1]} = u\}$$

is called **cylinder**. The family of these sets is a base of the induced topology on X. When it will not create confusion we will write $[u]$ and T instead of $[u]_X$ and $T_{|X}$.

Let x be a sequence on A and $\Omega(\mathsf{x})$ be the set $\{\mathsf{y} \in A^{\mathbb{N}}; \mathsf{y}_{[i,j]} \in L(\mathsf{x}), \forall\ [i,j] \subset \mathbb{N}\}$. It is clear that $(\Omega(\mathsf{x}), T)$ is a subshift. We say that $(\Omega(\mathsf{x}), T)$ is the subshift generated by x. Notice that $\Omega(\mathsf{x}) = \overline{\{T^n\mathsf{x}; n \in \mathbb{N}\}}$. Let (X, T) be a subshift on A, the following are equivalent:

(1) (X, T) is minimal.
(2) For all $\mathsf{x} \in X$ we have $X = \Omega(\mathsf{x})$.
(3) For all $\mathsf{x} \in X$ we have $L(X) = L(\mathsf{x})$.

We also have that $(\Omega(\mathsf{x}), T)$ is minimal if and only if x is uniformly recurrent. Note that if (Y, T) is another subshift then, $L(X) = L(Y)$ if and only if $X = Y$.

2.4. **Substitution subshifts.** Let us start with an important result for this lecture.

Proposition 3. *Let σ be a substitution. The sequence $(\sigma^n(aa\cdots))_{n\in\mathbb{N}}$ converges in $A^{\mathbb{N}}$ to a sequence x. The substitution σ is continuous on $A^{\mathbb{N}}$ and x is a* **fixed point** *of σ, i.e $\sigma(\mathsf{x}) = \mathsf{x}$.*

Proposition 4. *Let σ be a primitive substitution. All the fixed points of σ are uniformly recurrent and generate the same minimal subshift, we call it the* **substitution subshift generated by** *σ and we denote it (X_σ, T).*

A subshift generated by a substitutive sequence is called *substitutive subshift*. In fact there is no fundamental difference between these two notions of subshifts as we have the following result.

Proposition 5. *Let (X, T) be a minimal subshift. The following statements are equivalent.*

(1) (X, T) is isomorphic to a primitive substitution subshift.
(2) (X, T) is isomorphic to a minimal substitutive subshift.
(3) (X, T) is isomorphic to a substitution subshift.

There is another way to generate subshifts. Let L be a language on the alphabet A and define $X_L \subset A^{\mathbb{N}}$ to be the set of sequences $\mathsf{x} = (\mathsf{x}_n)_{n\in\mathbb{N}}$ such that each word of $L(\mathsf{x})$ appears in a word of L. The pair (X_L, T) is a subshift and we call it the **subshift generated by** L.

Proposition 6. *If σ is a primitive substitution, then $X_\sigma = X_L$ where $L = \{\sigma^n(a); a \in A, n \in \mathbb{N}\}$.*

It is easy to show that if x is an ultimately periodic sequence and $(\Omega(\mathsf{x}), T)$ is minimal, then x is periodic.

2.5. **Return words and linearly recurrent sequences.** For the rest of the section x is a uniformly recurrent sequence on the alphabet A and (X, T) is the minimal subshift it generates. We recall that all sequences in X are uniformly recurrent. Let u be a non-empty word of $L(X)$.

A word w on A is a **return word to** u in x if there exist two consecutive occurrences j, k of u in x such that $w = \mathsf{x}_{[j,k)}$. The set of return words to u is denoted by $\mathcal{R}_u(\mathsf{x})$. It is immediate to check that a word $w \in A^+$ is a return word if and only if:

(1) $uwu \in L(\mathsf{x})$ (i.e. uwu is a factor of x);
(2) u is a prefix of wu;
(3) the word wu has only two occurrences of u.

Let us make the following observations.

(1) As x is uniformly recurrent, the difference between two consecutive occurrences of u in x is bounded, and the set $\mathcal{R}_u(\mathsf{x})$ of return words to u is finite.
(2) The previous statement (2) cannot be simplified: it is not equivalent to u *is a prefix of* w. For example, if aaa is a factor of x then the word a is a return word to aa.
(3) From this characterization, it follows that the set of return words to u is the same for all $y \in X$, hence we set $\mathcal{R}_u(X) = \mathcal{R}_u(\mathsf{x})$.

If it is clear from the context, we write $\mathcal{R}_u$ instead of $\mathcal{R}_u(\mathsf{x})$.

We say that a sequence x on a finite alphabet is **linearly recurrent (for the constant** $K \in \mathbb{N}$**)** if it is recurrent and if, for every word u of x and all $w \in \mathcal{R}_u$ it holds

$$|w| \leq K|u|.$$

Proposition 7. *Let* $\mathsf{x} \in A^{\mathbb{N}}$ *be an non-periodic linearly recurrent sequence for the constant* K*. Then:*

(1) The number of distinct factors of length n *of* x *is less or equal to* Kn*.*
(2) x *is* $(K+1)$*-power free (i.e.* $u^{K+1} \in L(\mathsf{x})$ *if and only if* $u = \epsilon$*).*
(3) For all $u \in L(\mathsf{x})$ *and for all* $w \in \mathcal{R}_u$ *we have* $(1/K)|u| < |w|$*.*
(4) For all $u \in L(\mathsf{x})$*,* $\#\mathcal{R}_u \leq K(K+1)^2$*.*

Proof. We start with a remark. Let n be a positive integer and $u \in L(\mathsf{x})$ a word of length $(K+1)n-1$. Let $v \in L(\mathsf{x})$ be a word of length n. The difference between two successive occurrences of v is smaller than Kn, consequently u has at least one occurrence of v. We have proved that: For each n, every words of length n has at least one occurrence in each word of length $(K+1)n-1$. From this remark we deduce (1).

Let $u \in L(\mathsf{x})$ be a word such that $u^{K+1} \in L(\mathsf{x})$. Each factor of x of length $|u|$ occurs in u^{K+1}. But in u^{K+1} occurs at most $|u|$ distinct factors of length $|u|$ of x. This contradicts the non-periodicity of x. (We recall that if for some n a sequence $\mathsf{y} \in A^{\mathbb{N}}$ has at most n different words of length n, then it is ultimately periodic, see [Morse and Hedlund 1938].)

Assume there exist $u \in L(\mathsf{x})$ and $w \in \mathcal{R}_u$ such that $|u|/K \geq |w|$. The word w is a return word to u therefore u is a prefix of wu. We deduce that w^K is a prefix of u. Hence w^{K+1} belongs to $L(\mathsf{x})$ because wu belongs to $L(\mathsf{x})$. Consequently $w = \epsilon$ and (3) is proved.

Let u be a factor of x and $v \in L(\mathsf{x})$ be a word of length $(K+1)^2|u|$. Each word of length $(K+1)|u|$ occurs in v, hence each return word to u occurs in v. It follows from (3) that in v will occur at most $K(K+1)^2|u|/|u| = K(K+1)^2$ return words to u, which proves (4). □

We say (X,T) is a **linearly recurrent subshift** if it is a minimal subshift that contains a linearly recurrent sequence. It is easy to check that, then, all elements of (X,T) are linearly recurrent.

3. Some useful properties of the substitutions: linear recurrence and existence of word's frequencies

In this section we develop the tools we will use to prove Theorem 1 and Corollary 2. For a substitution $\tau : A^* \to A^*$, we set

$$|\sigma| = \max\{|\sigma(a)|; a \in A\} \text{ and } \langle\sigma\rangle = \min\{|\sigma(a)|; a \in A\}.$$

3.1. **Desubstitution.** Let σ be a substitution and (X, T) be the subshift it generates. Let u be a word of the language $L(X)$. From Proposition 6, v is a factor of some $\sigma^n(a)$. Thus there exist u_1, v_0 and w_0 belonging to $L(X)$, such that the length of v_0 and w_0 are less than $|\sigma|$ and $v = v_0\sigma(u_1)w_0$. Proceeding like this with u_1 and so on one obtains there exist n, and, words u_n, v_i $(0 \leq i \leq n-1)$ and w_i $(0 \leq i \leq n-1)$ such that

(1) $|u_n|$, $|v_i|$ and $|w_i|$, $0 \leq i \leq n-1$ are less that where $|\sigma|$;
(2) v_n is non-empty;
(3) $u = v_0\sigma(v_1)\cdots\sigma^{n-1}(v_{n-1})\sigma^n(u_n)\sigma^{n-1}(w_{n-1})\cdots(w_1)w_0$.

Observe that the v_i and the w_i can be empty.

3.2. **Linear recurrence of primitive substitutive sequences.** In this section we show that all fixed points of primitive substitutions are linearly recurrent.

Lemma 8. *Let σ be a primitive substitution. There exists a constant C such that*

$$|\sigma^k| \leq C\langle\sigma^k\rangle.$$

Moreover, if the incidence matrix of σ has positive entries then one can take $C = |\sigma|$.

Proof. For all k we choose some letters a_k and b_k such that $|\sigma^k(a_k)| = \langle\sigma^k\rangle$ and $|\sigma^k(b_k)| = |\sigma^k|$. By primitivity there exists k_0 such that for all $a, b \in A$ the letter b has an occurrence in the word $\sigma^{k_0}(a)$. We set $C = |\sigma^{k_0}(b_{k_0})|$. For $k \geq k_0$ we have

$$|\sigma^k| = \left|\sigma^k(b_k)\right| = \left|\sigma^{k_0}\left(\sigma^{k-k_0}(b_k)\right)\right| \leq C\left|\sigma^{k-k_0}(b_k)\right| \leq C\left|\sigma^k(a_k)\right| = C\langle\sigma^k\rangle.$$

□

Proposition 9. *All primitive substitutive sequences, and the subshifts they generate, are linearly recurrent.*

Proof. It suffices to prove it for fixed points of primitive substitutions. Let σ be a primitive substitution and x one of its fixed points. Without loss of generality one can suppose the incidence matrix of σ has positive entries. In virtue of Lemma 8 one has $|\sigma^k| \leq |\sigma|\langle\sigma^k\rangle$ for all k.

For all k we choose some letters a_k and b_k such that $|\sigma^k(a_k)| = \langle\sigma^k\rangle$ and $|\sigma^k(b_k)| = |\sigma^k|$. By primitivity there exists k_0 such that for all $a, b \in A$ the letter b has an occurrence in the word $\sigma^{k_0}(a)$. We set $C = |\sigma^{k_0}(b_{k_0})|$. For $k \geq k_0$ we have

$$|\sigma^k| = \left|\sigma^k(b_k)\right| = \left|\sigma^{k_0}\left(\sigma^{k-k_0}(b_k)\right)\right| \leq C\left|\sigma^{k-k_0}(b_k)\right| \leq C\left|\sigma^k(a_k)\right| = C\langle\sigma^k\rangle.$$

Let u be a word of $L(\mathsf{x})$ and w be a return word to u. Let k be the smallest integer such that $I_k \geq |u|$. The choice of k entails that there exists a word $ab \in L(\mathsf{x})$ of length 2 such that u occurs in $\sigma^k(ab)$. Let R be the largest difference between two successive occurrences of a word of length 2 of $L(\sigma)$. It follows

$$|w| \leq R|\sigma^k| \leq RC\langle\sigma^k\rangle \leq RCS_1\langle\sigma^{k-1}\rangle \leq RC|\sigma||u| \ .$$

□

When the substitution σ is primitive we will also say that σ is linearly recurrent for some constant.

3.3. **Perron Theorem and frequencies of the letters.** The following well-known theorem is fundamental to prove the existence and compute the frequencies of the words in fixed points of substitutions. The proof can be found in [Horn and Johnson 1990] or [Lind and Marcus 1995].

Theorem 10. *Let M be a $d \times d$ primitive matrix. Then :*

(1) *The matrix M has a positive eigenvalue θ which is strictly greater than the modulus of any other eigenvalue;*
(2) *The eigenvalue θ is algebraically simple;*
(3) *To this eigenvalue corresponds an eigenvector with positive entries.*
(4) *There exist $0 < r < \theta$ and C such that for all $i, j \in \{1, \ldots, d\}$ and all $n \in \mathbb{N}$ we have*

$$\left|M^n_{ij} - r_i l_j \theta^n\right| \leq Cr^n,$$

where $(r_1, \ldots, r_A)$ and $(l_1, \ldots, l_A)$ are respectively the unique right and left eigenvectors satisfying

$$\sum_{a \in A} r_a = 1 \text{ and } \sum_{a \in A} r_a l_a = 1. \tag{1}$$

Let $\sigma : A^* \to A^*$ be a primitive substitution, M its matrix and x one of its fixed points. The eigenvalue θ of the previous theorem will be called the **Perron eigenvalue** of M or σ. These real numbers are called **Perron numbers**. We take the notations of the previous theorem.

For a word $u \in L(\mathsf{x})$ we call **frequency** of u in $L(\mathsf{x})$ the limit (when it exists)

$$\text{freq}_\sigma(u) = \lim_{|v| \to \infty, v \in L(\mathsf{x})} \frac{1}{|v|} \# \left\{0 \leq i \leq |v| - |u| - 1; u = v_{[i,i+|u|-1]}\right\} . \tag{2}$$

We recall that for all n and all a, b in A we have $|\sigma^n(b)|_a = (M^n)_{a,b}$. Consequently from Perron theorem we obtain, for all $n \in \mathbb{N}$,

$$||\sigma^n(b)|_a - r_a l_b \theta^n| \leq Cr^n, \text{ thus} \tag{3}$$

$$||\sigma^n(b)| - l_b\theta^n| \leq (\#A)Cr^n \text{ and} \tag{4}$$

$$||\sigma^n(b)|_a - r_a|\sigma^n(b)|| \leq C(1 + \#A)r^n. \tag{5}$$

We set $C' = C(1 + \#A)$. We fix $a \in A$. Now we prove that $\text{freq}_\sigma(a)$ exists. Let $v \in L(\sigma)$. There exist n, and, words v_i $(0 \leq i \leq n)$ and w_i $(0 \leq i \leq n)$ such that

(1) $|v_i| \leq L$ and $|w_i| \leq L$ for $0 \leq i \leq n$ where $L = \max_{b \in A} |\sigma(b)|$;
(2) v_n is non-empty;
(3) $v = v_0\sigma(v_1) \cdots \sigma^{n-1}(v_{n-1})\sigma^n(v_n)\sigma^{n-1}(w_{n-1}) \cdots (w_1)w_0$.

Moreover from (4) there exists a constant $C'' > 0$ such that $|\sigma^n(u)| \geq C''|u|\theta^n$ for all $u \in L(\sigma)$. Hence

$$||v|_a - r_a|v|| \leq 2C' \frac{r^{n+1}-1}{r-1} \leq C'''|v|^\alpha,$$

for some constants C'', C''', where $\alpha = \log r / \log \theta < 1$. This means that $\mathrm{freq}_\sigma(a)$ exists and is equal to r_a.

3.4. **Substitutions of the words of length n and frequencies of words.** Here $\sigma : A^* \to A^*$ is a primitive substitution. In this section we prove the frequency of words exists for primitive substitutions. We use the previous subsection and the following substitutions.

Let $k \geq 1$. We consider $A_k = \{(u); u \in L(\sigma), |u| = k\}$ as an alphabet, the map $\pi_k : A_k^* \to A^*$ defined, for all $(u) = (u_1 \dots u_k) \in A_k$, by $\pi_k((u)) = u_1$, and the substitution $\sigma_k : A_k^* \to A_k^*$ in the following way: For $(u) \in A_k$ with $\sigma(u) = v = v_1 \dots v_m$ and $p = |\sigma(u_1)|$, we put

$$\sigma_k((u)) = (v_{[1,k]})(v_{[2,k+1]}) \cdots (v_{[p,p+k-1]}).$$

In other words, $\sigma_k(u)$ consists of the ordered list of the first $|\sigma(u_1)|$ factors of length k of $\sigma(u)$. Let $\mathsf{x} = x_0 x_1 \cdots$ be a fixed point of σ. Then, one can prove by induction that

$$(\sigma_k)^n((\mathsf{x}_{[0,k-1]})) = (\mathsf{x}_{[0,k-1]})(\mathsf{x}_{[1,k]}) \cdots (\mathsf{x}_{[|\sigma^n(x_0)|-1, |\sigma^n(x_0)|+k-1]}). \tag{6}$$

Notice that, for every $n > 1$, $(\sigma^n)_k$ is associated to σ^n in the same way as σ_k is associated to σ: $(\sigma^n)_k((u))$ consists of the ordered list of the first $|\sigma^n(u_1)|$ factors of length k of $\sigma^n(u)$. In particular we have:

$$|\sigma_k^n((u))| = |\sigma^n(u_1)|. \tag{7}$$

Thus from (6) we clearly see that $(\sigma_k)^n = (\sigma^k)_n$. For sake of simplicity we prefer to write σ_k^n.

If n is large enough, every $v \in L(\sigma)$ of length k is a factor of $\sigma^n(a)$ for every $a \in A$; Thus, $(v) \in A_k$ occurs in $\sigma_k^n((u))$ for every $(u) \in A_k$. We proved that σ_k is primitive.

Let w be a word of length $n > 0$ over the alphabet A_k. From the definition of $L(\sigma)$ and $L(\sigma_k)$ it can be checked that : $w \in L(\sigma_k)$ if and only if there exists a word $v \in L(\sigma)$ of length $n + k - 1$ such that $w = (v_{[1,k]})(v_{[2,k+1]}) \dots (v_{[n,n+k-1]})$. Clearly, given $(u) \in A_k$, the number of occurrences of the symbol (u) in w is equal to the number of occurrences of u in v. Consequently

$$\mathrm{freq}_\sigma(u) = \mathrm{freq}_{\sigma_k}((u)). \tag{8}$$

Finally applying the results of the previous subsection to σ_k for all k we obtain the following result.

Proposition 11. *For all $u \in L(\sigma)$ there exist constants $\mathrm{freq}_\sigma(u)$, D and $\alpha < 1$ such that for all $v \in L(\sigma)$ of length greater than $|u|$ we have*

$$||v|_u - \mathrm{freq}_\sigma(u)\,(|v| - |u| + 1)| \leq D\,(|v| - |u| + 1)^\alpha.$$

Corollary 12. *The frequency of u exists for all $u \in L(\sigma)$.*

In [Holton and Zamboni 1999] is proved the following theorem which is central in the present paper. Below we give a different proof using Proposition 7.

Theorem 13. *Let θ be the Perron eigenvalue of σ. There exists a finite set $F \subset \mathbb{R}$ such that for all $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ satisfying*

$$\{\mathrm{freq}_\sigma(u); u \in L(\sigma), |u| = n\} \subset \left\{s\theta^{-k}; s \in F\right\}.$$

Proof. We suppose σ is linearly recurrent for the constant K. Let θ_2 be the Perron eigenvalue of σ_2. From (7) and Perron Theorem we deduce that $\theta_2 = \theta$. Let M_2 be the incidence matrix of σ_2. From (8) we know that $\mathrm{freq}_\sigma(u) = \mathrm{freq}_{\sigma_2}((u))$ for all $u \in A_2$. Subsection 3.3 and Theorem 10 imply that $(\mathrm{freq}_{\sigma_2}((u)); (u) \in A_2)$ is the unique right eigenvector of M_2 (for the eigenvalue θ_2) with $\sum_{(u)\in A_2} \mathrm{freq}_{\sigma_2}((u)) = 1$.

Let C be the constant defined in Lemma 8. Let $u \in L(\sigma)$ be a word of length n and k an integer satisfying such that

$$\langle \sigma^{k-1} \rangle \leq |u| = n \leq \langle \sigma^k \rangle$$

Let B be the set of words $(ab) \in A_2$ such that u has an occurrence in $\sigma^k(ab)$. The choice of k implies this set is non-empty. Let $(ab) \in B$ and $M = \max\{|\sigma(c)|, c \in A\}$. From Proposition 7 it follows that

$$|\sigma^k(ab)|_u \leq \frac{|\sigma^k(ab)|}{|u|/K} \leq \frac{2K|\sigma|C\langle\sigma^{k-1}\rangle}{|u|} \leq 2K|\sigma|C.$$

Moreover from (4) we have that

$$\lim_{m\to\infty} \frac{|\sigma^{m+k}(a)|}{|\sigma^m(a)|} = \theta^k$$

and from Proposition 11, for all $c \in A$,

$$\mathrm{freq}_\sigma(u) = \lim_{m\to\infty} \frac{|\sigma^{m+k}(c)|_u}{|\sigma^{m+k}(c)|}.$$

Let $a'b'$ be the last word of length two of $\sigma^m(c)$. Then,

$$\begin{aligned}
\frac{|\sigma^{m+k}(c)|_u}{|\sigma^{m+k}(c)|} &= \frac{\sum_{ab\in A_2}\left(|\sigma^k(ab)|_u - |\sigma^k(b)|_u\right)|\sigma^m(c)|_{ab} + |\sigma^k(b')|_u}{|\sigma^{m+k}(c)|} \\
&= \sum_{ab\in A_2}\left(|\sigma^k(ab)|_u - |\sigma^k(b)|_u\right)\frac{|\sigma^m(c)|_{ab}}{|\sigma^m(c)|}\frac{|\sigma^m(c)|}{|\sigma^{m+k}(c)|} + \frac{|\sigma^k(b')|_u}{|\sigma^{m+k}(c)|} \\
&\longrightarrow_{m\to\infty} \sum_{ab\in A_2}\left(|\sigma^k(ab)|_u - |\sigma^k(b)|_u\right)\mathrm{freq}_\sigma(ab)\theta^{-k}.
\end{aligned}$$

Consequently, it suffices to take

$$F = \left\{\sum_{ab\in A_2} j_{ab}\mathrm{freq}_\sigma(ab); j_{ab} \in [0, 2K|\sigma|C] \cap \mathbb{N}, ab \in A_2\right\}$$

which is a finite set. □

4. Cobham's theorem for minimal substitutive systems

In this section we prove Theorem 1 and Corollary 2.

4.1. **Preimages of factor maps of LR subshifts.** Let ϕ be a factor map from the subshift (X,T) on the alphabet A onto the subshift (Y,T) on the alphabet B. If there exists a r**-block map** $f : A^{2r+1} \to B$ such that $(\phi(x))_i = f(x_{[i-r,i+r]})$ for all $i \in \mathbb{N}$ and $x \in X$, we shall say that f is a **block map associated to** ϕ, that f **defines** ϕ and that ϕ is a **sliding block code**. The theorem of Curtis-Hedlund-Lyndon (Theorem 6.2.9 in [Lind and Marcus 1995]) asserts that factor maps are sliding block codes.

Theorem 14 (Curtis-Hedlund-Lyndon theorem)**.** *Let ϕ be a factor map between two subshifts. Then, there exists a block map f associated to ϕ.*

Proof. We left it as an exercise. □

If $u = u_0u_1 \cdots u_{n-1}$ is a word of length $n \geq 2r+1$ we define $f(u)$ by $(f(u))_i = f(u_{[i,i+2r]})$, $i \in \{0, 1, \cdots, n-2r-1\}$.

Let C denote the alphabet A^{2r+1} and $Z = \{((x_{[-r+i,r+i]}); i \in \mathbb{N}) \in C^{\mathbb{N}}; (x_n; n \in \mathbb{N}) \in X\}$. It is easy to check that the subshift (Z,T) is isomorphic to (X,T) and that f induces a 0-block map from C onto B which defines a factor map from (Z,T) onto (Y,T).

The next lemma was first proved in [Durand 2000].

Lemma 15. *Let (X,T) be a non-periodic LR subshift (for the constant K) and (Y,T) be a non-periodic subshift factor of (X,T). Then (Y,T) is LR. Moreover, there exists n_1 such that: For all $u \in L(Y)$ with $|u| \geq n_1$ we have*

(1) $|u|/2K \leq |w| \leq 2K|u|$ for all $w \in \mathcal{R}_u(Y)$;
(2) $\#(\mathcal{R}_u(Y)) \leq 2K(2K+1)^2$.

Proof. We denote by A the alphabet of X and by B the alphabet of Y. Let $\phi : (X,T) \to (Y,T)$ be a factor map. Let $f : A^{2r+1} \to B$ be a block map associated to ϕ.

Let u be a word of $L(Y)$ and $v \in L(X)$ be such that $f(u) = v$. We have $|u| = |v| - 2r$. If w is a return word to u then $|w| \leq \max\{|s|; s \in \mathcal{R}_v\} \leq K|v| \leq K(|u|+2r)$. Then, the subshift (Y,T) is linearly recurrent for the constant $K(2r+1)$. Moreover: For all $u \in L(Y)$ such that $|u| \geq n_1 = 2r$, and for all $w \in \mathcal{R}_u$, $|w| \leq 2K|u|$. To obtain the other inequality it suffices to proceed as in the proof of Proposition 7.

Let $u \in L(Y)$ with $|u| \geq n_1$ and $v \in L(Y)$ be a word of length $(2K+1)^2|u|$. Each word of length $(2K+1)|u|$ occurs in v, hence each return word to u occurs in v. It follows from the previous assertion that in v occurs at the most $2K(2K+1)^2|u|/|u| = 2K(2K+1)^2$ return words to u. □

Proposition 16. *Let (X,T) be a non-periodic LR subshift (for the constant K). Let $\phi : (X,T) \to (Y,T)$ be a factor map such that (Y,T) is a non-periodic subshift and $f : A^{2r+1} \to B$ be a r-block map defining ϕ. Then there exists n_0 such that for all $u \in Y$, with $|u| \geq n_0$, we have*

$$\#(f^{-1}(\{u\})) \leq 4K(K+1).$$

Proof. Let n_1 be the integer given by Lemma 15. We set $n_0 = \max(2r+1, n_1)$. Let $u \in L(Y)$ such that $|u| \geq n_0$. The difference between two distinct occurrences of elements of $f^{-1}(\{u\})$ is greater than $|u|/2K$. Moreover $f^{-1}(\{u\})$ is included in $L(X) \cap A^{|u|+2r}$ and each word of length $(K+1)(|u|+2r)$ has an occurrence of each word of $L(X) \cap A^{|u|+2r}$. Therefore

$$\#(f^{-1}(\{u\})) \leq \frac{(K+1)(|u|+2r)}{|u|/2K} \leq 4K(K+1).$$

This completes the proof. □

4.2. **Frequencies in the factors.** Let σ be a primitive substitution with dominant eigenvalue α and linearly recurrent constant K, and, (Y,T) a non-periodic factor of (X_σ, T). Let $\phi : X_\sigma \to Y$ be a factor map and f be a r-block map that defines ϕ. From Theorem 13 we know there exists a finite set $F_\sigma \subset \mathbb{R}$ such that for all n there exists $k \in \mathbb{N}$ satisfying

$$\{\mathrm{freq}_\sigma(v); v \in L(X_\sigma), |v| = n\} \subset \left\{s\theta^k; s \in F_\sigma\right\}. \tag{9}$$

Let $u \in L(Y)$, $|u| = m$, and set $f^{-1}(\{u\}) = \{v_1, \ldots, v_l\} \subset L_{|u|+2r}(X_\sigma)$ with $l \leq 4K(K+1)$ (Proposition 16). Let k be as in (9) for $n = |u| + 2r$. Let $y \in Y$ and $x \in X_\sigma$ such that $\phi(x) = y$. We remark that

$$\lim_{|v|\to\infty, v \in L(Y)} \frac{1}{|v|} \# \left\{0 \leq i \leq |v| - |u|; u = v_{[i,i+|u|-1]}\right\}$$

exists and is equal to

$$\lim_{|w|\to\infty, w \in L(X)} \frac{1}{|w|} \# \left\{0 \leq i \leq |w| - |u| + 2r; w_{[i,i+|u|+2r-1]} \in \{v_1, \ldots, v_l\}\right\}$$

We denote it $\mathrm{freq}_Y(u)$. Moreover,

$$\mathrm{freq}_Y(u) = \sum_{i=1}^{l} \mathrm{freq}_{X_\sigma}(v_i) \in \left\{s'\theta^k; s' \in F'_\sigma\right\},$$

where F'_σ is the finite set $\left\{\sum_{i=1}^{4K(K+1)} f_i; f_i \in F_\sigma, 1 \leq i \leq 4K(K+1)\right\}$. We proved:

Theorem 17. *Let θ be the Perron eigenvalue of the primitive substitution σ. There exists a finite set $F \subset \mathbb{R}$ such that for all non-periodic subshift factor (Y,T) of (X_σ, T), and all $n \in \mathbb{N}$ there exists $k \in \mathbb{N}$ satisfying*

$$\{\mathrm{freq}_Y(u); u \in L(Y), |u| = n\} \subset \left\{f\theta^k; f \in F\right\}.$$

4.3. **Proof of Theorem 1.** From Theorem 17 we know there exist two finite sets F_σ and F_τ such that for all $n \in \mathbb{N}$ there exists $k, k' \in \mathbb{N}$ satisfying

$$\{\mathrm{freq}_Y(u); u \in L(Y), |u| = n\} \subset \left\{s\alpha^k; s \in S_\sigma\right\} \cap \left\{s\beta^{k'}; s \in S_\tau\right\}.$$

Using Statement (1) of Lemma 15 we have that $\mathrm{freq}_Y(u) \leq 2K/|u|$ for all $u \in L(Y)$. Then, $\mathrm{freq}_Y(u)$ tends to 0 when $|u|$ goes to infinity. Consequently, there exist $u, v \in L(Y)$, $s \in S_\sigma$, $t \in S_\tau$, $k, k' \in \mathbb{N}$, $k \neq k'$, and $l, l' \in \mathbb{N}$, $l \neq l'$, such that

$$s\alpha^k = \mathrm{freq}_Y(u) = t\beta^l \text{ and } s\alpha^{k'} = \mathrm{freq}_Y(v) = t\beta^{l'}.$$

We obtain that $\alpha^{k'-k} = \beta^{l'-l}$, which ends the proof.

5. Measure theoretical dynamical systems and ergodicity

The goal of this section is to translate some results we obtained on frequencies of words into results on shift invariant measures. To this end we need to enlarge the framework to measurable dynamical systems.

A **measurable dynamical system** is a quadruple $(X, \mathcal{B}, \mu, T)$, where X is a space endowed with a σ-algebra $\mathcal{B}$, a probability measure μ and measurable map $T : X \to X$ that preserves the measure μ, *i.e.*, $\mu(T^{-1}B) = \mu(B)$ for any $B \in \mathcal{B}$. We also say that μ is T-invariant. The measure μ is called **ergodic** if any T-invariant measurable set has measure 0 or 1. If (X, S) admits a unique measure preserved by S, then the system is said **uniquely ergodic**.

It is well-known that a dynamical system (X, T) endowed with the Borel σ-algebra always admits a probability measure μ preserved by the map T, and then form a measurable dynamical system.

Theorem 18 (Krylov-Bogolioubov theorem)**.** *Let (X, T) be a dynamical system. There exists at least one Borel probability measure on X preserved by T.*

Proof. Let $M(X)$ be the set of Borel probability measures on X endowed with the weak-star topology. Fix an arbitrary point of X and consider a weak-star cluster point μ in $M(X)$ of the sequence of probability measures $(\mu_N)_N$:

$$\mu_N = \frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^n x},$$

where δ_y stands for the Dirac measure at y. Clearly, μ is a probability measure preserving which is T-invariant. □

The most important result for measurable dynamical system is the **Ergodic theorem**. We refer to [Queffélec 2010] or [Petersen 1983] for more details.

Theorem 19 (Birkhoff Ergodic Theorem)**.** *Let $(X, \mathcal{B}, \mu, T)$ be a measurable dynamical system. Let $f \in L^1(X)$. Then the sequence*

$$\left(\frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n \right)_{N \geq 0}$$

converges μ-almost everywhere to a function $f^* \in L^1(X)$. One has $f^* \circ T = f^*$ μ-almost everywhere and $\int_X f^* d\mu = \int_X f d\mu$. Furthermore, if T is ergodic, then f^* is μ-almost everywhere constant and for all f belonging to $L^1(X)$ on has, for μ-almost every x,

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ T^n(x) = \int_X f d\mu.$$

The Ergodic Theorem applied to characteristic χ_C functions of cylinders C can be restated in terms of frequencies. Let us defined for a subshift X, when it exists, the frequencies of $u \in L(X)$ in the sequence x:

$$\mathrm{freq}_X(\mathsf{x}, u) = \lim_{n \to \infty} \frac{1}{n} \# \left\{ 0 \leq i \leq n - |u| - 1; u = \mathsf{x}_{[i, i+|u|-1]} \right\}.$$

We say (X,S) has **uniform frequencies** whenever $\text{freq}_X(\mathsf{x},u)$ exists for all u and x We have seen in Corollary 12 primitive substitution subshifts have uniform frequencies. Below we show that uniform frequencies property characterize unique ergodicity following the presentation of S. Ferenczi and T. Monteil in their Chapter 7 in [Durand 2010].

Proposition 20. *Let (X,S) be a subshift and μ be an ergodic measure on (X,S). Then for μ-almost every $\mathsf{x} \in X$, and for any word $w \in L(X)$, the frequency $\text{freq}_X(\mathsf{x},v)$ exists and is equal to $\mu([w])$.*

Proof. It suffices to apply the Ergodic Theorem to the characteristic functions of cylinder sets $f = \chi_{[w]}$ noticing that

$$|\mathsf{x}_{[0,N-1]}|_w = \sum_{n=0}^{N-1} \chi_{[w]} \circ S^n(\mathsf{x}).$$

□

Proposition 21. *Let (X,T) be a uniquely ergodic dynamical system whose unique invariant measure is denoted by μ. Let $f : X \to \mathbb{R}$ be a continuous function. Then, the sequence of functions $(\frac{1}{N}\sum_{n=0}^{N-1} f \circ T^n)_{N\geq 0}$ converges uniformly to the function with constant value $\int_X f d\mu$.*

Proof. We proceed by contradiction: there exist $\epsilon > 0$, a sequence $(x_n)_{n\in\mathbb{N}}$ and a strictly increasing integer sequence α such that for any integer n:

$$\frac{1}{\alpha(n)} \sum_{k=0}^{\alpha(n)-1} f(T^k(x_{\alpha(n)}) - \int_X f d\mu \geq \epsilon.$$

Let ν be a cluster point of the sequence of probability measures (ν_n) defined by

$$\nu_n = \frac{1}{\alpha(n)} \sum_{k=0}^{\alpha(n)-1} \delta_{T^k(x_{\alpha(n)})}.$$

The measure ν is T-invariant and satisfies

$$\int_X f d\nu - \int_X f d\mu \geq \epsilon.$$

This contradicts the uniqueness of μ. □

Corollary 22. *Subshifts have uniform frequencies if and only if they are uniquely ergodic.*

Proof. This is a consequence of propositions 21 and 20 □

6. Exercises

6.1. **Exercises of Section 1.**

Exercise 1. Show that $E \subset \mathbb{N}$ is a finite union of arithmetic progressions if and only if its characteristic sequence $(x_n)_n$ ($x_n = 1$ if $n \in E$ and0 otherwise) is equal to $uvvv\cdots$ for some words u, v.

6.2. **Exercises of Section 2.**

Exercise 2. Let $\tau_1 : B^* \to C^*$, $\tau_2 : A^* \to B^*$ and $\tau_3 : A^* \to C^*$ be three morphisms such that $\tau_1 \circ \tau_2 = \tau_3$, then we have the following equality: $M_{\tau_1} M_{\tau_2} = M_{\tau_3}$.

Exercise 3. Show that the triadic Cantor space is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

Exercise 4. Show that any compact set X having a countable basis of its topology which consists of closed and open sets and does not have isolated points is homeomorphic to $\{0,1\}^{\mathbb{N}}$.

Exercise 5. Show that the distance given by

$$d(\mathsf{x},\mathsf{y}) = \frac{1}{2^n} \quad \text{with} \quad n = \inf\{|k|; \mathsf{x}_k \neq \mathsf{y}_k\},$$

where $\mathsf{x} = (\mathsf{x}_n; n \in \mathbb{N})$ and $\mathsf{y} = (\mathsf{y}_n; n \in \mathbb{N})$ are two elements of $A^{\mathbb{N}}$, defines the topology of $A^{\mathbb{N}}$.

Exercise 6. Show that the shift transformation is continuous.

Exercise 7. Let (X,S) be a dynamical system. Show that there exists a minimal subsystem, that is a dynamical system $(Y, S_{/Y})$ where Y is a compact subset of X (Hint: use Zorn's Lemma).

Exercise 8. Let x be a sequence on A and $\Omega(\mathsf{x})$ be the set $\{\mathsf{y} \in A^{\mathbb{N}}; \mathsf{y}_{[i,j]} \in L(\mathsf{x}), \forall\ [i,j] \subset \mathbb{N}\}$. Show that $(\Omega(\mathsf{x}), T)$ is a subshift.

Exercise 9. Let (X,T) be a subshift on the alphabet A, the following are equivalent:

(1) (X,T) is minimal.
(2) For all $\mathsf{x} \in X$ we have $X = \Omega(\mathsf{x})$.
(3) For all $\mathsf{x} \in X$ we have $L(X) = L(\mathsf{x})$.

Exercise 10. Show that $(\Omega(\mathsf{x}), T)$ is minimal if and only if x is uniformly recurrent.

Exercise 11. Let (X,T) and (Y,T) be two subshifts. Show that $L(X) = L(Y)$ if and only if $X = Y$.

Exercise 12. Prove propositions 3, 4 and 6

Exercise 13. Let x be a uniformly recurrent. Show the following.

(1) The difference between two consecutive occurrences of u in x is bounded, and the set $\mathcal{R}_u(\mathsf{x})$ of return words to u is finite.
(2) The set of return words to u is the same for all $y \in \Omega(x)$.

Exercise 14. Let x be a linearly recurrent sequence. Show that all $\mathsf{y} \in \Omega(\mathsf{x})$ are linearly recurrent.

Exercise 15. Let x be a fixed point of a primitive substitution on the alphabet A. Let $\phi : A^* \to B^*$ be a morphism. Show that $\phi(x)$ is linearly recurrent.

Exercise 16. Show that the primitive substitution σ defined by $0 \mapsto 0010$ and $1 \mapsto 1$ has two fixed points and that both are linearly recurrent.

6.3. **Exercises of Section 3.**

Exercise 17. Show that the dominant eigenvalue of a primitive substitution σ of constant length, that is $|\sigma| = \langle\sigma\rangle$, is $|\sigma|$.

Exercise 18. Compute the frequencies of the words of length 1, 2 and 3 of the substitutions

$$\sigma: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 10 \end{array} \quad \tau: \begin{array}{l} 0 \mapsto 01 \\ 1 \mapsto 0 \end{array} \quad \xi: \begin{array}{l} 0 \mapsto 012 \\ 1 \mapsto 010 \\ 2 \mapsto 221. \end{array}$$

Exercise 19. Let σ be a primitive substitution. Express matricial relations between the incidence matrices of σ and its substitutions of the words of length n. Deduce some properties on the eigenvalues.

6.4. **Exercises of Section 4.**

Exercise 20. Prove the Curtis-Hedlund-Lyndon theorem

Exercise 21. Consider the two subshifts generated by the following (non primitive) substitutions:

$$\sigma: \begin{array}{l} 0 \mapsto 0121 \\ 1 \mapsto 1112 \\ 1 \mapsto 2111 \end{array} \qquad \xi: \begin{array}{l} 0 \mapsto 012 \\ 1 \mapsto 112 \\ 2 \mapsto 211. \end{array}$$

Show they are non isomorphic.

Exercise 22. Prove the substitutive version of Cobham's theorem in the primitive case.

Exercise 23. Prove the substitutive version of Cobham's theorem.

6.5. **Exercises of Section 5.**

Exercise 24. Let (X, S) be a primitive substitution subshift. Prove that $\mathrm{freq}_X(x, u)$ exists for all $x \in X$ and $u \in L(X)$ and is independent of x.

References

[Allouche and Shallit 2003]
J.-P. Allouche and J. O. Shallit. *Automatic Sequences, Theory, Applications, Generalizations.* Cambridge University Press, 2003.

[Cobham 1969]
A. Cobham. On the base-dependence of sets of numbers recognizable by finite automata. *Math. Systems Theory* **3** (1969), 186–192.

[Cobham 1972]
A. Cobham. Uniform tag sequences. *Math. Systems Theory* **6** (1972), 164–192.

[Durand 2000]
F. Durand. Linearly recurrent subshifts have a finite number of non-periodic subshift factors. *Ergodic Theory Dynam. Systems* **20** (2000), 1061–1078.

[Durand 2010]
F. Durand. Infinite words and invariant measures. In V. Berthé and M. Rigo, editors, *Combinatorics, Automata and Number Theory*, Vol. 135 of *Encyclopedia of Mathematics and its applications*, pp. 338–386. Cambridge Univ. Press, 2010.

[Eilenberg 1974]
S. Eilenberg. *Automata, Languages, and Machines*, Vol. A. Academic Press, 1974.

[Holton and Zamboni 1999]
C. Holton and L. Q. Zamboni. Descendants of primitive substitutions. *Theory Comput. Systems* **32** (1999), 133–157.

[Horn and Johnson 1990]
R. A. Horn and C. R. Johnson. *Matrix analysis*. Cambridge University Press, Cambridge, 1990. Corrected reprint of the 1985 original.

[Lind and Marcus 1995]
D. Lind and B. Marcus. *An Introduction to Symbolic Dynamics and Coding*. Cambridge University Press, 1995.

[Morse and Hedlund 1938]
M. Morse and G. A. Hedlund. Symbolic dynamics. *Amer. J. Math.* **60** (1938), 815–866.

[Pansiot 1986]
J.-J. Pansiot. Decidability of periodicity for infinite words. *RAIRO Inform. Théor. App.* **20** (1986), 43–46.

[Petersen 1983]
K. Petersen. *Ergodic theory*. Cambridge University Press, 1983.

[Queffélec 2010]
Martine Queffélec. *Substitution dynamical systems—spectral analysis.Second Edition*, Vol. 1294 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.

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