RIGIDITY RESULTS IN CELLULAR AUTOMATA THEORY: PROBABILISTIC AND ERGODIC THEORY APPROACH

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Abstract. In these notes review some results and its extensions concerning the existence of invariant stationary probability measures under a one-dimensional *algebraic* cellular automaton. We present two historical axes of this question and the techniques used to solve them or produce relevant intermediate results. Both make appear strong *rigidity* phenomena, i.e. the unique solution is the uniform Bernoulli product measure. The first axe is the ergodic theory approach where we impose some natural conditions on the entropy and ergodicity of the system to get the result. This approach follows ideas by Rudolph [18] and Host [9] in the classical problem called $(\times 2, \times 3)$ in the circle posed by Hillel Furstenberg at the end of the 60'. Then we present a purely probabilistic approach. We study the convergence of the Cesàro mean of the iterates by an algebraic cellular automaton of a translation invariant probability measure. Assuming some natural correlation properties (this class includes Markov and Gibbs probability measures of full topological support) one proves the limit exists and it is the uniform Bernoulli product measure.

Contents

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1. Introduction

The study of the dynamics of cellular automata is often associated to its capacity to model complex systems using very simple local interactions in the phase space. From the probabilistic point of view one possible interpretation of such complex behavior is the richness of the space of invariant probability measures and the existence of limit measures for the iterates of interesting classes of probability measures by cellular automata. Such initial conditions can be thought as the law that we use to produce random configurations in the phase space of the cellular automata. Both questions represent major challenges of the theory of cellular automata and few results exist even in the one-dimensional setting.

A remarkable situation appears when considering classes of algebraic cellular automata. That is, when in addition, the underlying phase space has an algebraic structure compatible with the cellular automata. In this context, it has been observed that even if the sets of invariant probability measures are rich, under very natural conditions the unique invariant measure is the maximal entropy one. From the perspective of iterating probability measures by cellular automata of algebraic origin these kinds of results lead to think that good candidates to be limits of such sequences of iterations are the same class of measures. In fact, such limits rarely exist being the Cesàro means of such sequences the good candidates to converge. Indeed, it has been observed by D. Lind in his pioneer work [12] that the Cesàro mean of the iterates of a Bernoulli measure on $\{0,1\}^{\mathbb{Z}}$ by the cellular automaton $F = \sigma^{-1} + \sigma$ converges to the uniform product measure $(\frac{1}{2}, \frac{1}{2})^{\mathbb{Z}}$. In this example σ is the shift map and $\{0,1\}^{\mathbb{Z}}$ is the product Abelian group with addition modulo two componentwise. Since Lind's results, this *rigid* phenomenon has being proved to be very general in the algebraic context and seems to be extensible to the context of positively expansive or expansive cellular automata. The purpose of these notes is to present what in the opinion of the author are the most illustrative results concerning the described phenomena in the case of algebraic cellular automata acting on a fullshift $A^{\mathbb{Z}}$ that at the same time is a compact Abelian group.

The notes are organized as follows. Section 2 is devoted to a minimum of background in symbolic dynamics and ergodic theory. In section 3 we present a basic example to illuminate the problems in a relevant *study case*. The rigidity phenomenon that appears in the set of invariant measures is presented in section 4. There we show that different ergodicity conditions together with some entropy conditions imply that the Haar measure is the unique invariant measure for the shift and the cellular automaton simultaneously. Section 5 is devoted to the study of the convergence of the Cesàro means of the iterates of nice classes of probability measures by algebraic cellular automata.

2. Preliminaries and Background

2.1. **Symbolic Dynamics (in dimension 1).** In this section we summarize the main background in Symbolic Dynamics that we will need in the article. For a more detailed exposition we suggest the book by D. Lind and B. Marcus [11].

Let A be a finite set or *alphabet*. Following this last nomenclature its elements are also called *letters* or *symbols*. Denote by A^* the set of finite sequences or *words* $w = w_0...w_{n-1} \in A^n$ with letters in A including the empty word ε. By $|w|$ we mean the length of $w \in A^*$ and $|\varepsilon| = 0$.

Let $X = A^{\mathbb{Z}}$ be the set of two-sided sequences

$$\mathbf{x} = (x_i)_{i\in\mathbb{Z}} = (\dots x_{-i} \dots x_0 \dots x_i \dots)$$

with symbols in A. Analogously one defines $X = A^{\mathbb{N}}$ to be the set of one-sided sequences in A. Both are called *full-shifts*. For simplicity we restrict our attention to the two-sided case. The space X is compact for the product topology and metrizable. A classical distance is given by:

$$d(\mathbf{x}, \mathbf{y}) = 2^{-\inf\{|i| \,:\, i\in\mathbb{Z},\ x_i\neq y_i\}},$$

for any $\mathbf{x}, \mathbf{y} \in X$, i.e. two points are close if they coincide in big windows near the origin. For $\mathbf{x} \in X$ and $i \leq j$ in $\mathbb{Z}$ or $\mathbf{x} = x_0 \dots x_n \in A^*$ and $i \leq j$ in $\{0, \dots, n\}$, we denote by $x[i, j] = x_i \dots x_j$ the finite word in $\mathbf{x}$ between coordinates i and j. Given $w \in A^*$ and $i \in \mathbb{Z}$, the cylinder set starting in coordinate i with word w is $[w]_i = \{\mathbf{x} \in X : x[i, i + |w| - 1] = w\}$.

A natural dynamical system on X is the *shift map* $\sigma : X \to X$, where $\sigma(\mathbf{x}) = (x_{i+1})_{i\in\mathbb{Z}}$. This map is a homeomorphism of X. If we need to distinguish a shift map according to its alphabet we denote it by σ_A.
We call $Y \subseteq X$ a *subshift* if it is closed (for the product topology) and $\sigma(Y) = Y$ (invariant for the shift map). A simple example is given by the orbit closure of a point $\mathbf{x}$ in X, i.e. $\overline{\{\sigma^n(\mathbf{x}) \ : \ n \in \mathbb{Z}\}}$. A special class of subshifts are *shifts of finite type*; they are inspired in Markov chains in probability theory. A subshift $Y \subseteq X$ is a subshift of finite type if there is a finite subset $\mathcal{W}$ of words in A of a given length L such that for any $\mathbf{y} \in Y$ and $i \in \mathbb{Z}$,

$$y_i \dots y_{i+L-1} \notin \mathcal{W}.$$

Example 2.1. Let $A = \{0, 1\}$ and consider $\mathcal{W} = \{11\}$. The subshift of finite type Y defined by $\mathcal{W}$ as described before consists of two-sided infinite sequences in A do not containing two consecutive ones.

The *language* of a subshift $Y \subseteq X$ is given by

$$\mathcal{L}(Y) = \{y[i,j] : \mathbf{y} \in Y,\ i,j \in \mathbb{Z},\ i \le j\}.$$

One saids that Y is *mixing* if for any $u, v \in \mathcal{L}(Y)$ there is $N \in \mathbb{N}$ such that for any $n > N$, $[u]_0 \cap \sigma^{-n}[v]_0 \neq \emptyset$. In other words, there is a word $w \in A^n$ such that $uwv \in \mathcal{L}(Y)$.
A second kind of important dynamics are given by continuous and shift commuting maps of a subshift $Y \subseteq X$. A map $F : Y \to Y$ is called a *block-map* if F is continuous (for the product topology) and $F \circ \sigma = \sigma \circ F$. They are called block maps since Hedlund's result [8] states that there is a *local map* $f : A^{m+a+1} \to A$, where $a, m \in \mathbb{N}$ (a is called anticipation and m memory), such that $\forall\ i \in \mathbb{Z}, \forall\ \mathbf{y} \in Y$

$$F(\mathbf{y})_i = f(y_{i-m}, \dots, y_{i+a}).$$

We also use f to indicate the action of the local rule on words of length greater than or equal to $m+a+1$. That is, for $w = w_0 ... w_n \in A^*$ with $|w| \ge m + a + 1$, we put

$$f(w) = f(w_0, ..., w_{m+a}) f(w_1, ..., w_{m+a+1}) ... f(w_{n-(m+a)}, ..., w_n).$$

If $m = 0$ or $a = 0$ one says that F is *one-sided.*

2.2. **Cellular automata and invariant measures.** Let $Y \subseteq A^{\mathbb{Z}}$ be a *mixing* shift of finite type and $F : Y \to Y$ a block-map. Then F is called a *cellular automaton* (CA). Typical examples correspond to $Y = A^{\mathbb{Z}}$ (a full-shift). Analogously one defines cellular automata acting on the set of one-sided subshifts $Y \subseteq A^{\mathbb{N}}$ but this case is not considered in these notes.

Let $Y \subseteq A^{\mathbb{Z}}$ be a subshift and $F : Y \to Y$ a block-map. A probability measure μ defined on the Borel σ-algebra of Y (we simply say "on Y") is F-*invariant* if for any Borel set B of Y

$$F\mu(B) := \mu(F^{-1}(B)) = \mu(B).$$

If $F = \sigma$ (the shift map on Y), the measure is said to be *stationary* or *shift invariant*. An invariant measure is *ergodic* if invariant Borel sets have measure 0 or 1. We observe that if μ is shift invariant, since F commutes wit the shift, then $F^n\mu$ is also shift invariant, where F^n is the n-th iterate of F.
In this paper we study the convergence of the *Cesàro mean* $\mathcal{M}_\mu^N(F) = \frac{1}{N}\sum_{n=0}^{N-1} F^n\mu$. If this limit exists as $N \to \infty$, we denote it by $\mathcal{M}_\mu(F)$. If $Y = A^{\mathbb{Z}}$ a main role will be played by the uniform product or Bernoulli measure $\lambda_A^{\mathbb{Z}}$ of $A^{\mathbb{Z}}$, where λ_A is the equidistributed probability measure on A.
The following classes of CA on $A^{\mathbb{Z}}$ are relevant for these notes:

1) *Linear CA.* Let $(A, +)$ be a finite Abelian group. This structure naturally extends to $A^{\mathbb{Z}}$ by componentwise operations, so $(A^{\mathbb{Z}}, +)$ is also an Abelian group (to simplify notations we also denote the operation by $+$). Observe that $(A^{\mathbb{Z}}, +)$ is a compact Abelian group and the uniform Bernoulli measure is its *Haar measure*. It is characterized as the unique probability measure μ such that $\mu(\chi) = \int_{A^{\mathbb{Z}}} \chi d\mu = 0$ for every non-trivial character $\chi \in \widehat{A^{\mathbb{Z}}}$, i.e. for $\chi \neq 1$.
A cellular automaton $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is said to be *linear* if for any $\mathbf{x} \in A^{\mathbb{Z}}$

$$F(\mathbf{x}) = \sum_{i=1}^{l} k_i \sigma^{n_i}(\mathbf{x})$$

where $n_1, \ldots, n_l, k_1, \ldots, k_l \in \mathbb{Z}$. In terms of the local rule this means:

$$F(\mathbf{x})_j = \sum_{i=1}^{l} k_i x_{j+n_i} \ .$$

2) *Permutative CA.* The CA $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is said to be *right permutative* if for every $w \in A^{m+a}$ the map $f(w, \cdot) : A \to A$ defined by $f(w, \cdot)(\alpha) = f(w\alpha)$ is one-to-one. This implies that for every $w \in A^{m+a}$ the map $f_w : A^{m+a} \to A^{m+a}$ given by $f_w(w') = f(ww')$ is also one-to-one. Analogously we define *left permutative* and *bipermutative* CA.

2.3. **Entropy.** A classical measure of complexity of the dynamics of a (classically surjective) CA F on a subshift $Y \subseteq A^{\mathbb{Z}}$ with respect to an F invariant measure μ is the *Shannon entropy* or just the *entropy*. It is defined as follows: for any $N \in \mathbb{N}$ let $\alpha_{N,\infty}$ be the σ-algebra given by $\bigvee_{n\geq 1} F^{-n}\alpha_N$, where $\alpha_N = \{\mathbf{y}[-N,N] : \mathbf{y} \in Y\}$ is a partition of Y, and put:

$$h_\mu(F) = -\lim_{N\to\infty} \sum_{C\in\alpha_N} \int_Y 1_C(\mathbf{y}) \log\left(\mathbb{E}(1_C|\alpha_{N,\infty})(\mathbf{y})\right) d\mu(\mathbf{y}) \ .$$

A probablity measure of *maximal entropy* (for the CA) is one for which:

$$h_\mu(F) = \sup_\nu h_\nu(F)$$

where the sup is taken over all F invariant probability measures on Y.

3. Main Questions

3.1. **Questions and comments.** Let $F : Y \to Y$ be a surjective cellular automaton on a mixing shift of finite type Y. In this article we assume $Y = A^{\mathbb{Z}}$. Here we set the three main questions of this theory.

Question 1: Study the set of *invariant measures* of F and in addition of the joint action of F and σ. That is, find probability measures μ on Y such that for any Borel set B and integers $n \in \mathbb{N}, m \in \mathbb{Z}$

$$F^n\mu(B) := \mu(F^{-n}B) = \mu(B)$$

or

$$F^n \circ \sigma^m\mu(B) := \mu(F^{-n} \circ \sigma^{-m}B) = \mu(B).$$

A natural invariant measure for F is the one of *maximal entropy* for the shift map. In fact, it holds that F is surjective if and only if the maximal entropy measure is also F-invariant [4]. Depending on the subshift Y and the dynamical properties of F it is possible to construct other invariant measures (see for example [26]). Nevertheless, in some cases strong rigidities appear, that is, this is the unique shift and F invariant probability measure.

Question 2: Given a shift invariant probability measure μ on Y study if the limit of the sequence $(F^n\mu : n \in \mathbb{N})$ exists (in the weak topology). We remark that every weak limit of a subsequence is invariant for F and the shift. It is also interesting the convergence when $N \to \infty$ of the Cesàro mean

$$\mathcal{M}_\mu^N(F) = \frac{1}{N}\sum_{n=0}^{N-1} F^n\mu \ .$$

One says F *asymptotically randomizes* μ if the limit of the Cesàro mean converges to the maximal entropy measure.

Question 3: Find conditions to ensure the maximal entropy measure is the unique solution to Questions 1 and 2.

Comments:

— In relation with Question 1 the type of solutions we look for are like the $(\times 2, \times 3)$ Furstenberg's problem in $\mathbb{R}/\mathbb{Z}$ [18]: F (or σ) is ergodic and σ (resp. F) has positive entropy for the invariant measure. While ergodicity of one transformation can be changed for a weaker condition the positivity of the entropy cannot be dropped for the moment. Proofs strongly rely on entropy formulas. These conditions already appear in Rudolph's or Host's solutions to $(\times 2, \times 3)$ problem and all recent improvements (see [9, 18]).

— In relation with Question 2 in the linear case there are two points of view. One is to consider measures μ of increasing complexity in correlations: Markov, Gibbs, other chain connected measures; then represent them as "independent processes" and prove that the limit of the Cesàro mean converges to the uniform Bernoulli product measure on $A^{\mathbb{Z}}$ [6]. The other one is motivated by Lind's work [12] and uses harmonic analysis. The idea is to define a class of *mixing* measures such that the Cesàro mean of the iterates of any of them converges.

– From Glasner and Weiss results in topological dynamics (see [7]) one gets that either the CA map F is *almost equicontinuous* or *sensitive to initial conditions*, and in the last class most interesting known examples (and in fact coming from Nasu's reductions [17]) are *expansive* or *positively expansive maps.* In the equicontinuous case or systems with equicontinuous points, orbits tend to be periodic and invariant measures can be more or less described but are not nice. Moreover, in this case the limits of the Cesàro means we are considering always converge [2]. If the CA are positively expansive they are *conjugate with shifts of finite type* (see [1, 3]), so we have two commuting shifts of finite type with the same maximal entropy measure. In this last case there can still exist an *equicontinuous direction* so invariant measures are as in the previous case.

– Good examples: the classes of positively expansive or expansive CA without equicontinuous directions have not been described. The main examples with these features correspond to *algebraic maps*, in particular linear CA. This is why most of the results concerning Questions 1, 2 and 3 are concentrated on these maps.

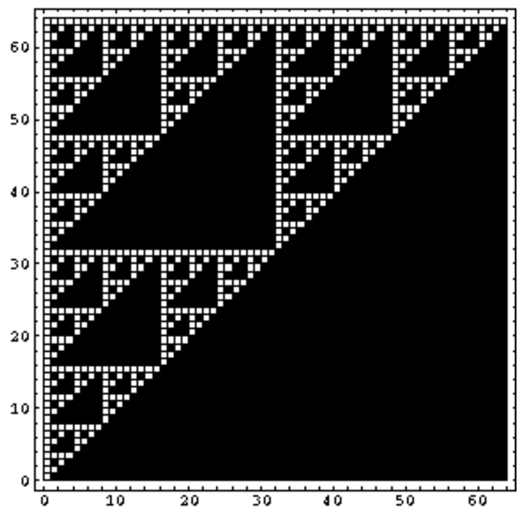

FIGURE 1. The iteration of a ultimately periodic configuration by the addition modulo 2 CA. One observes the Pascal triangle modulo 2 behind it.

3.2. **Basic example: addition modulo 2 or *Ledrappier's three dot problem.*** Let $X = \{0,1\}^{\mathbb{Z}}$ and see X as an Abelian group with coordinatewise addition modulo 2. Let $F : X \to X$ be given by $F = id + \sigma$, where σ is the shift map on X. That is, $F(\mathbf{x})_i = x_i + x_{i+1}$. Remark that it is a 2-to-1 surjective map.

– **In relation to Question 1:** Natural invariant measures for F and the shift map simultaneously are the uniform Bernoulli product measure $\lambda = (1/2, 1/2)^{\mathbb{Z}}$ and measures supported on periodic orbits for F and the shift, but other invariant measures of algebraic origin has been described (see [26]).

– **In relation to Question 2:** In general the limit does not exist. It follows from a good understanding of the Pascal triangle modulo 2.
We give a brief argument in the Bernoulli case. Let $\mu = (\pi_0, \pi_1)^{\mathbb{Z}}$ be a Bernoulli non-uniform product measure on X with $\pi_0 = \mu(x_i = 0)$,

$\pi_1 = \mu(x_i = 1)$. A simple induction yields to:

$$\mu\left(\sum_{i\in I} x_i = a\right) = \frac{1}{2}\left(1 + (-1)^a(\pi_0 - \pi_1)^{\#I}\right)$$

where I is a finite subset of $\mathbb{N}$. Thus,

$$F^n\mu([a]_0) = \mu\left(\sum_{k\in I(n)} x_k = a\right) = \frac{1}{2}\left(1 + (-1)^a(\pi_0 - \pi_1)^{\#I(n)}\right)$$

where $I(n) = \{0 \leq k \leq n \ : \ \binom{n}{k} = 1 \bmod 2\}$.
If $a = 0$, for the subsequence along $n = 2^m$ the limit exists and is equal to $\pi_0^2 + \pi_1^2$ and for the subsequence along $n = 2^m - 1$ the limit is $\frac{1}{2}$. But the Cesàro mean converges:

$$\mathcal{M}_\mu^N(F)([a]_0) = \frac{1}{N}\sum_{n=0}^{N-1} F^n\mu([a]_0) = \frac{1}{2} + \frac{(-1)^a}{2}\frac{1}{N}\sum_{n=0}^{N-1}(\pi_0 - \pi_1)^{\#I(n)}$$

since $\lim_{N\to\infty} \frac{\{0\leq n<N \ : \ \#I(n)\geq\alpha\log\log N\}}{N} = 1$ for some $\alpha \in (0, 1/2)$ (a non trivial consequence of Lucas' lemma, see [6]) then the limit is $\frac{1}{2}$. Also, using similar arguments, one proves that $\mathcal{M}_\mu^N(F)([a_0 \ldots a_{l-1}]_0)$ converges to $\frac{1}{2^l}$ as $N \to \infty$. This was observed by D. Lind in 84 for $F = \sigma^{-1} + \sigma$ [12].

This result reinforces the idea that the uniform Bernoulli product measure $\lambda = (1/2, 1/2)^{\mathbb{Z}}$ must be the unique invariant measure of F and σ verifying *some conditions to be determined*. The following sections will go deeper on these conditions and proofs will be drafted for this particular but relevant example.

4. Ergodic approach: results on invariant measures for a CA and the shift simultaneously

4.1. **A theorem for the basic example.** The *model* theorem in the theory concerns the basic example. Most of the existing generalizations start from this example changing the precise local map, the cardinality of the alphabet or the algebraic structure of $A^{\mathbb{Z}}$. In the rest of this section $\{0, 1\}^{\mathbb{Z}}$ is seen as a product Abelian group where addition modulo 2 is applied componentwise.

Theorem 4.1 (Basic Theorem [10])**.** *Let $F : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be the CA given by $F = id + \sigma$. If μ is an F and σ invariant probability measure on $\{0,1\}^{\mathbb{Z}}$ with $h_\mu(F) > 0$ and is ergodic for σ then $\mu = \lambda = (1/2, 1/2)^{\mathbb{Z}}$.*

Sketch of the proof. Let μ be a simultaneously invariant probability measure for F and the shift σ. Put $X = \{0,1\}^{\mathbb{Z}}$. We describe the main steps of the proof, the difficulty is not homogeneous but each step only requires a little computation:

1) Let $\mathcal{B}$ be the product σ-algebra of X and $\mathcal{B}_1 = F^{-1}\mathcal{B}$. For μ–a.e. $\mathbf{x} \in X$ define $\mu_{\mathbf{x}}(\cdot) = \mathbb{E}(\cdot|\mathcal{B}_1)(\mathbf{x})$. This measure is concentrated on $\{\mathbf{x}, \mathbf{x}+\mathbf{1}\}$, where $\mathbf{1} = \ldots 1111111 \ldots \in X$. Also, $\sigma\mu_{\mathbf{x}} = \mu_{\sigma\mathbf{x}}$.
2) Define $\phi(\mathbf{x}) = \mu_{\mathbf{x}}(\{\mathbf{x}+\mathbf{1}\})$. Then

$$\phi \circ \sigma(\mathbf{x}) = \mu_{\sigma\mathbf{x}}(\{\sigma\mathbf{x}+\mathbf{1}\}) = \sigma\mu_{\mathbf{x}}(\{\sigma\mathbf{x}+\mathbf{1}\}) = \mu_{\mathbf{x}}(\{\mathbf{x}+\mathbf{1}\}) = \phi(\mathbf{x})$$

3) The **ergodicity** of μ with respect to σ implies that ϕ is constant μ–a.e., thus also $F\mu$-a.e. This implies that

$$\phi \circ F = \phi \circ \sigma = \phi, \ \mu - \text{a.e.} \qquad (*)$$

4) Define $E = \{\mathbf{x} \in X \ : \ \phi(\mathbf{x}) > 0\}$.

– If $B \subseteq X$ is the *good set* of measure one where $(*)$ is satisfied then $\mu\{\mathbf{x} \in E : \mathbf{x}+\mathbf{1} \in B\} = \mu(E)$;

– so, $\phi(\mathbf{x}+\mathbf{1}) = \phi(F(\mathbf{x}+\mathbf{1})) = \phi(F(\mathbf{x})) = \phi(\mathbf{x})$ for μ-a.e. $\mathbf{x}$ in E;

– that is, $\mu_{\mathbf{x}}(\{\mathbf{x}\}) = \mu_{\mathbf{x}}(\{\mathbf{x}+\mathbf{1}\}) = \frac{1}{2}$ for μ-a.e. $\mathbf{x}$ in E.

5) E is σ-invariant by (*), then by **ergodicity** $\mu(E) = 0$ or $\mu(E) = 1$.

6) **Entropy formula:** let $\alpha = \{[0]_0, [1]_0\}$. Therefore, using standard computations in the entropy theory of dynamical systems one deduces (see for instance [19]):

$$h_\mu(F) = -\sum_{a \in \{0,1\}} \int_X 1_{[a]_0}(\mathbf{x}) \log\left(\mathbb{E}(1_{[a]_0}|\mathcal{B}_1)(\mathbf{x})\right) d\mu(\mathbf{x})$$

Observe that when $\mathbf{x} \in [a]_0$ then $\mu_{\mathbf{x}}([a]_0) = \mu(\{\mathbf{x}\})$ since $\mathbf{x}+\mathbf{1} \notin [a]_0$ for $a = 0, 1$. Then

$$h_\mu(F) = -\int_X \log(\mu_{\mathbf{x}}(\{\mathbf{x}\}))d\mu(\mathbf{x}) \ .$$

7) Conclusion:

– If $h_\mu(F) > 0$ then from 6) one deduces that $\mu(E) > 0$. Therefore, from 5) (**ergodicity**) follows that $\mu(E) = 1$;

– this last fact implies by 4) that: $\mu_{\mathbf{x}}(\{\mathbf{x}\}) = \mu_{\mathbf{x}}(\{\mathbf{x}+\mathbf{1}\}) = \frac{1}{2}$ for μ-a.e. $\mathbf{x} \in X$;

– concluding by 6) that $h_\mu(F) = \log(2)$. Since $\lambda = (1/2, 1/2)^{\mathbb{Z}}$ is the unique maximal entropy measure for F (or similarly is the unique stationary probability measure on X verifying last equality) then $\mu = \lambda$.

□

4.2. **Some generalizations.** As commented before, several generalizations can be expected. The next two are of different nature. The first one consists just in changing $\{0,1\}^{\mathbb{Z}}$ by $\mathbb{Z}_p = \{0, \ldots, p-1\}^{\mathbb{Z}}$ where p is a prime number. Its proof is essentially copying the one given before. In the second one, the ergodicity condition is weaker, then we need to add a condition on the σ-algebra of invariant sets. Here, we do not change the local rules, but all reasonable linear extensions follows also directly.

Theorem 4.2 (Host, Maass, Martínez, [10])**.** *Let $F : \mathbb{Z}_p^{\mathbb{Z}} \to \mathbb{Z}_p^{\mathbb{Z}}$ be given by $F = id + \sigma$. Let μ be an F and σ invariant probability measure on $\mathbb{Z}_p^{\mathbb{Z}}$. If $h_\mu(F) > 0$ and μ is ergodic for the shift then μ is the uniform product measure $(1/p, \ldots, 1/p)^{\mathbb{Z}}$.*

We need an additional concept. One says that an invariant probability measure μ for a CA $F : Y \to Y$ and the shift map on Y is (F, σ)-ergodic if any Borel set B in Y that is invariant for the joint action of such maps has measure 0 or 1, i.e., if $\mu(F^{-n}\sigma^{-m}B \Delta B) = 0$ for any $n \in \mathbb{N}, m \in \mathbb{Z}$ then $\mu(B) = 0$ or $\mu(B) = 1$. We denote by $\mathcal{I}_\mu(F) = \{B \in \mathcal{B}(Y) : \mu(F^{-1}B\Delta B) = 0\}$, the set of invariant Borel sets for F with respect to μ.

Theorem 4.3 (Host, Maass, Martínez, [10])**.** *Let $F : \mathbb{Z}_p^{\mathbb{Z}} \to \mathbb{Z}_p^{\mathbb{Z}}$ be given by $F = id + \sigma$. Let μ be an F and σ invariant probability measure on $\mathbb{Z}_p^{\mathbb{Z}}$. If $h_\mu(F) > 0$, μ is (F, σ)-ergodic and $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{p(p-1)})$, then μ is the uniform product measure $(1/p, \ldots, 1/p)^{\mathbb{Z}}$.*

The next generalizations try to extract from previous theorems those properties that seems to be the main objects behind this class of results. We say a CA $F : Y \to Y$ is *algebraic* if Y (in addition to be a mixing shift of finite type) is a compact Abelian topological group and F and the shift are endomorphisms of such group. Here, the role of the uniform product measure is played by the Haar measure of the compact Abelian group (the unique probability measure that is invariant by translation by elements of the group). An invariant measure for F is said to be *totally ergodic* if it is ergodic for all powers of F. This is a very strong ergodicity condition that allows frequently in this theory to jump over the difficult obstacles. It implies the technical hypothesis about invariant σ-algebras in Theorem 4.3, but this last is a much more refined condition.

Theorem 4.4 (Pivato, [23]). *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an algebraic bipermutative CA. If μ is a totally ergodic invariant probability measure for σ, $h_\mu(F) > 0$ and $Ker(F)$ has no shift invariant subgroups, then μ is the Haar measure of $A^{\mathbb{Z}}$.*

The most general extension of Theorem 4.3 not using total ergodicity is the following.

Theorem 4.5 (Sablik, [25]). *Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an algebraic bipermutative CA and Σ be an F and σ invariant closed subgroup of $A^{\mathbb{Z}}$. Fix $k \in \mathbb{N}$ such that any prime divisor of $|A|$ divides k. Let μ be an F and σ invariant probability measure on $A^{\mathbb{Z}}$ with $supp(\mu) \subseteq \Sigma$ such that:*
– μ is (F, σ)-ergodic;
– $h_\mu(F) > 0$;
– $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_1})$, where p_1 is the smallest common period of the elements in $Ker(F)$;
– any finite shift invariant subgroup of $\cup_{n\in\mathbb{N}} Ker(F^n) \cap \Sigma$ is dense in Σ.
Then μ is the Haar measure of Σ.

These theorems have an analogous result in a much more general algebraic context. We do not give all the details of this theory since they escape from the context of these notes. The main point is that, instead of considering the actions of a CA F and the shift map on a mixing shift of finite type Y, one considers d commuting actions on a 0-dimensional set. In our context $d = 2$, the commuting actions are F and σ and the 0-dimensional space is Y.

Theorem 4.6 (Einsiedler, [5]). *Let α be an algebraic $\mathbb{Z}^d$-action of a compact 0-dimensional Abelian group verifying some algebraic conditions. Let μ be an invariant (for the complete action) probability measure. Then, if the action has positive entropy in one direction and the measure is totally ergodic for the action then it is the Haar measure of the group.*

Remark 4.1. From last theorems it is possible to deduce the same kind of results for some classes of positively expansive and expansive CA actions on a fullshift, *a priori* not algebraic (see [10]).

5. Probabilistic approach: results on the convergence of Cesàro means

In this section we will present the main results concerning the convergence of Cesàro means of the iterates of a probability measure by algebraic cellular automata. In [12] D. Lind proposes an harmonic analysis point of view to study the convergence of the Cesàro means of the iterates of a Bernoulli product measure by the CA $\sigma^{-1} + \sigma$ on $\{0,1\}^{\mathbb{Z}}$ seen as a product Abelian group. This technique cannot work alone, it needs a fine combinatorial analysis of the Pascal triangle modulo 2. The extension of Lind's pioneer results to other classes of initial probability measures and other types of algebraic cellular automata was considered in [13] and was deepened in [6]. In these works the main example is $id + \sigma$ in $\mathbb{Z}_p^{\mathbb{Z}}$ where p is a prime number. The main issue was that Bernoulli measures were changed by probability measures with complete connections and summable decay of correlations, class that includes Markov and Gibbs measures for example. The harmonic analysis ideas used by Lind were not used here, the technical part was to represent such general class of measures using independent processes via regeneration idea. The harmonic analysis reappeared with the works of M. Pivato and R. Yassawi ([20, 21]). They put into the concept of *harmonically mixing* measures (introduced by them) the main properties observed in the classes of measures considered in [6], and the dynamical properties of the algebraic CA considered (that essentially comes from Pascal triangle) gave rise to the concept of *diffusivity*, giving an abstract formalization to previous results. The complexification of the classes of algebraic CA considered was achieved in several further works [22, 23, 24, 14, 15, 16]. For simplicity we will not describe such results in this review. The main statements there propose the same kind of results as in the basic (but fundamental) cases up to some natural and necessary technical conditions.

5.1. **Harmonic analysis point of view.** Let $(A,+)$ be a finite Abelian group and fix $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ an algebraic CA.
A character $\chi : A \to \mathbb{T}^1$ in $\widehat{A^{\mathbb{Z}}}$, where $\mathbb{T}^1$ is the one-dimensional torus, is given by $\chi = \bigotimes_{k\in\mathbb{Z}} \chi_k$ where χ_k are characters of A and $\chi_k = 1$ for all but finitely many terms in this product. The rank of the character χ, rank(χ), is the number of non trivial characters χ_k in $\bigotimes_{k\in\mathbb{Z}} \chi_k$.
The Haar or uniform Bernoulli measure λ on $A^{\mathbb{Z}}$ is characterized by

$$\lambda(\chi) = \int_{A^{\mathbb{Z}}} \chi d\lambda = 0 \quad \forall \chi \not\equiv 1 \ . \tag{1}$$

Definition 5.1 (Pivato, Yassawi, [20])**.** A probability measure μ on $A^{\mathbb{Z}}$ is *harmonically mixing* if $\forall \varepsilon > 0 \ \exists N(\varepsilon) > 0$ such that $\forall \chi \in \widehat{A^{\mathbb{Z}}}$:

$$\text{rank}(\chi) > N(\varepsilon) \Rightarrow |\mu(\chi)| = \left| \int_{A^{\mathbb{Z}}} \chi d\mu \right| < \varepsilon \ .$$

If $A = \mathbb{Z}_p$, then a Markov probability measure with strictly positive transitions is harmonically mixing.

Definition 5.2 (Pivato, Yassawi, [20])**.** • The block map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is *diffusive* if

$$\forall \chi \not\equiv 1 : \lim_{n\to\infty} \ \text{rank}\,[\chi \circ F^n] = \infty.$$

• F is *diffusive in density* if there exists $J \subseteq \mathbb{N}$ of density 1 such that

$$\lim_{\substack{n\to\infty \\ n\in J}} \ \text{rank}\,[\chi \circ F^n] = \infty.$$

The following theorem can be considered as a consolidation of results in [20, 21, 6]. Nevertheless, historically this form appeared first in [21].

Theorem 5.1. *Let $(A,+)$ be a finite Abelian group and $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an algebraic CA with local map given by $f(x_{i-m} \dots x_{i+a}) = \sum_{k=-m}^{a} f_k(x_{i+k})$, where $f_{-m}, \dots, f_a$ are commuting automorphisms of the group $(A,+)$ and at least two of them are non-trivial. Then F is diffusive in density and for any harmonically mixing probability measure μ on $A^{\mathbb{Z}}$:*

$$\mathcal{M}_\mu(F) = \lim_{N\to\infty} \mathcal{M}_\mu^N(F) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} F^n\mu = \lambda.$$

Sketch of the proof. The diffusitivy of F comes from the combinatorial structure of the associated generalized Pascal triangle and we do not give it here. The consequence on the convergence of the Cesàro means follows directly from the combination of diffusivity of F and the harmonically mixing condition of μ. Indeed, let χ be a non-trivial character in $\widehat{A^{\mathbb{Z}}}$ and $N \in \mathbb{N}$. A simple computation yields to,

$$\int_{A^{\mathbb{Z}}} \chi d\mathcal{M}_{\mu}^{N}(F) = \frac{1}{N}\sum_{n=0}^{N-1}\int_{A^{\mathbb{Z}}} \chi \circ F^n d\mu \ .$$

Assume F is diffusive in density and consider $J \subseteq \mathbb{N}$ with density 1 such that $\lim_{\substack{n\to\infty \\ n\in J}}$ rank $[\chi \circ F^n] = \infty$ and put $J_N = J \cap \{0, \ldots, N-1\}$. One gets,

$$\int_{A^{\mathbb{Z}}} \chi d\mathcal{M}_{\mu}^{N}(F) = \frac{1}{N}\sum_{n\in J_N}\int_{A^{\mathbb{Z}}} \chi \circ F^n d\mu + \frac{1}{N}\sum_{n\in J_N^c}\int_{A^{\mathbb{Z}}} \chi \circ F^n d\mu \ .$$

Thus,

$$\left|\int_{A^{\mathbb{Z}}} \chi d\mathcal{M}_{\mu}^{N}(F)\right| \leq \left|\frac{1}{N}\sum_{n\in J_N}\int_{A^{\mathbb{Z}}} \chi \circ F^n d\mu\right| + \frac{|J_N^c|}{N} \ .$$

Since μ is harmonically mixing, given $\epsilon > 0$ there is $N(\epsilon) \in \mathbb{N}$ such that for any $n \geq N(\epsilon)$ in J_N,

$$\left|\int_{A^{\mathbb{Z}}} \chi \circ F^n d\mu\right| \leq \epsilon \ .$$

One concludes, taking the limit as $N \to \infty$ that:

$$\lim_{N\to\infty}\int_{A^{\mathbb{Z}}} \chi d\mathcal{M}_{\mu}^{N}(F) \leq \epsilon \ .$$

By (1), this implies that $\mathcal{M}_{\mu}^{N}(F)$ converges and its limit is equal to the Haar measure λ. □

5.2. **Regeneration of measures point of view.** In this subsection we present the probabilistic approach proposed by P. Ferrari, A. Maass, S. Martínez and P.Ney in [6]. Further works that use the same idea are [10, 16].

Let μ be any shift invariant probability measure on a fullshift $A^{\mathbb{Z}}$ and consider $w = (\ldots, w_{-2}, w_{-1}) \in A^{-\mathbb{N}}$ (for our purposes $-\mathbb{N} = \{\ldots -4, -3, -2, -1\}$). We denote by μ_w the conditional probability measure on $A^{\mathbb{N}}$.

Definition 5.3. One says that μ has *complete connections* if given $a \in A$ and $w \in A^{-\mathbb{N}}$, $\mu_w([a]_0) > 0$. If μ is a probability measure with complete connections, one defines for every $m \geq 1$

$$\gamma_m = \sup \left(\left| \frac{\mu_v([a]_0)}{\mu_w([a]_0)} - 1 \right| \ : \ v, w \in A^{-\mathbb{N}};\ v_{-i} = w_{-i},\ 1 \leq i \leq m \right) .$$

In addition, if $\sum_{m \geq 1} \gamma_m < \infty$ one says μ has *summable decay of correlations.*

The main result in [6] states:

Theorem 5.2 (Ferrari, Maass, Martínez, Ney)**.** *Let $(A, +)$ be a finite Abelian group with $|A| = p^s$ with p prime. Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the CA given by $id + \sigma$. If μ is a probability measure on $A^{\mathbb{Z}}$ with complete connections and summable decay of correlations, then for all $w \in A^{-\mathbb{N}}$ it holds*

$$\mathcal{M}_\mu(F) = \lim_{N \to \infty} \mathcal{M}_\mu^N(F) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} F^n \mu = \lambda \ .$$

We observe that the technical conditions on the cardinality of A and the form of F can be dropped to state a result similar to that in Theorem 5.1. Historically this is the first large generalization (in relation to the class of initial measures) to Lind's result.

5.2.1. *Main Idea behind the proof: regeneration of measures.* Beyond the study of the combinatorial properties of the Pascal triangle modulo a prime number the main ingredient in the proof of Theorem 5.2 is the representation of a probability measure with complete connections and summable decay of correlations by means of uniform independent variables. The idea behind is trying to *mimic* the computations made in the Bernoulli case.

Let $(T_i : i \geq 1)$ be an increasing sequence of non-negative integer random variables. For every finite subset L of $\mathbb{N}$ define

$$\mathbf{N}(L) = \Big| \{ i \geq 1 : T_i \in L \} \Big| \ .$$

One says that $(T_i : i \geq 1)$ is a *stationary renewal process with finite mean interrenewal time* if

(1) $(T_i - T_{i-1} : i \geq 2)$ are independent identically distributed with finite expectation, they are independent of T_1 and $\mathbb{P}(T_2 - T_1 > 0) > 0$;

(2) For $n \in \mathbb{N}$, $\mathbb{P}(T_1 = n) = \frac{1}{\mathbb{E}(T_2 - T_1)} \mathbb{P}(T_2 - T_1 > n)$.

These conditions imply the stationary property: for every finite subset L of $\mathbb{N}$ and every $a \in \mathbb{N}$ the random variables $\mathbf{N}(L)$ and $\mathbf{N}(L+a)$ have the same distribution.

Theorem 5.3 (Ferrari, Maass, Martínez, Ney, [6]). *Let μ be a shift invariant probability measure on $A^{\mathbb{Z}}$ with complete connections and summable decay of correlations. There exists a stationary renewal process $(T_i : i \geq 1)$ with finite mean interrenewal time such that for every $w \in A^{-\mathbb{N}}$, there exists a random sequence $z = (z_i : i \geq 1)$ with values in A and distribution μ_w such that $(z_{T_i} : i \geq 1)$ are i.i.d. uniformly distributed in A and independent of $(z_i : i \in \mathbb{N} \setminus \{T_1, T_2, \ldots\})$.*

From the construction of the renewal process in [6] one also gets the following properties:

(1) There exists a function $\rho : \mathbb{N} \to \mathbb{R}$ decreasing to zero such that $\mathbb{P}(\mathbf{N}(L) = 0) \leq \rho(|L|)$, for any finite subset L of $\mathbb{N}$.

(2) Given $n, \ell \in \mathbb{N} \setminus \{0\}$, $1 \leq k_1 < ... < k_\ell \leq n$ and $j_1, ..., j_l \in \mathbb{N}$, for all $a_1, ..., a_n \in A$,

$$\mu_w\left(z_i = a_i, i \in \{1, ..., n\}; T_{j_1} = k_1, ..., T_{j_\ell} = k_\ell\right) =$$

$$\frac{1}{|A|^\ell}\mu_w\left(z_i = a_i, i \in \{1, ..., n\} \setminus \{k_1, ..., k_\ell\}; T_{j_1} = k_1, ..., T_{j_\ell} = k_\ell\right) .$$

(3) For any $n \in \mathbb{N}$ and $v \in A^*$, $\mu_w\left(\{\mathbf{N}(\{0, \ldots, n-1\}) > 0\} \cap [v]_n\right)$ does not depend on $w \in A^{-\mathbb{N}}$.

In the language of [20] these properties allow to prove that shift invariant probability measures with complete connections and summable decay of correlations on $A^{\mathbb{Z}}$ are harmonically mixing, and thus one can conclude Theorem 5.2. The proof in [6] did not follow this path explicitly, nevertheless they are analogous.

Theorem 5.4 (Host, Maass, Martínez, [10]; use ideas in [6]). *A shift invariant probability measure with complete connections and summable decay of correlations on $A^{\mathbb{Z}}$ is harmonically mixing.*

Sketch of the proof. Let μ be a shift invariant probability measure on $A^{\mathbb{Z}}$ with complete connections and summable decay of correlations. Fix a past sequence $w \in A^{-\mathbb{N}}$ and let $(T_i : i \geq 1)$ be the renewal process induced by μ. We write $\mathbb{P}_w$ for the probability, when the random variables $(z_i : i \in \mathbb{N})$ in Theorem 5.3 are given the distribution μ_w. The probability measure $\mathbb{P}$ is the integral of $\mathbb{P}_w$ with respect to $w \in A^{-\mathbb{N}}$.

For a finite subset R of $\mathbb{Z}$ and $\mathbf{x} \in A^{\mathbb{Z}}$ we write x_R for the sequence $(x_i : i \in R)$ in A^R.
Let $\chi : A^{\mathbb{Z}} \to \mathbb{T}^1$ be a character. There exist a finite set $R \subset \mathbb{Z}$ and a sequence $(\chi_n : n \in \mathbb{Z})$ in $\widehat{A}$, with $\chi_n = 1$ for $n \notin R$, $\chi_n \neq 1$ for $n \in R$ and $\chi(x) = \prod_{n\in\mathbb{Z}} \chi_n(x_n)$ for every $\mathbf{x} \in A^{\mathbb{Z}}$. We have to find an upper bound for $|\int \chi \, d\mu|$ depending only on $|R|$. As μ is shift invariant we can assume that $R \subset \mathbb{N}$.
For any finite subset R' of $\mathbb{Z}$ define $\chi_{R'} : A^{R'} \to \mathbb{T}^1$ by $\chi_{R'}(y) = \prod_{r\in R'} \chi_r(y_r)$. Observe that $\chi_{R'}(A^{R'})$ is a subgroup of $\mathbb{T}^1$ and that we can identify χ with χ_R because $\chi_R(x_R) = \chi(\mathbf{x})$ for $\mathbf{x} \in A^{\mathbb{Z}}$.
Denote $\Xi = \chi_R(A^R)$ and define $\tau(R) = \inf\{i \in R : \mathbf{N}(\{i\}) = 1\}$, where $\inf \emptyset = \infty$. We have

$$\int_{A^{\mathbb{Z}}} \chi(x_R) d\mu_w(\mathbf{x}) = \sum_{\xi\in\Xi} \xi \; \mu_w(\chi(x_R) = \xi)$$

$$= \sum_{i\in R} \sum_{\xi\in\Xi} \xi \, \mathbb{P}_w(\chi(z_R) = \xi, \tau(R) = i) + \sum_{\xi\in\Xi} \xi \, \mathbb{P}_w\big(\chi(z_R) = \xi, \tau(R) = \infty\big) \, .$$

Let $i \in R$ and set $R_i = R \setminus \{i\}$, so $\Xi = \chi_{R_i}(A^{R_i})\chi_i(A)$. For $\xi \in \Xi$ we define $V_i(\xi) = \chi_{R_i}^{-1}(\xi\chi_i(A))$. A word $y = (y_r : r \in R_i) \in A^{R_i}$ belongs to $V_i(\xi)$ if and only if there exists $a \in A$ such that the word y' obtained from y by putting $y'_i = a$ satisfies $\chi(y') = \xi$.
For $y \in A^{R_i}$ we put $\xi_y = \chi_{R_i}(y)$ and for $\xi \in \Xi$ we define $V_i(\xi, y) = \chi_i^{-1}(\xi\xi_y^{-1})$. Since $V_i(\xi, y)$ is a coset of $Ker(\chi_i)$, we get $|V_i(\xi, y)| = |Ker(\chi_i)|$. Therefore,

$$\begin{aligned}
&\sum_{\xi\in\Xi} \xi \; \mathbb{P}_w\big(\chi(z_R) = \xi, \tau(R) = i\big) \\
&\qquad = \sum_{\xi\in\Xi} \xi \sum_{y\in V_i(\xi)} \sum_{y_i\in V_i(\xi,y)} \mathbb{P}_w\big(z_r = y_r, \; r \in R; \tau(R) = i\big) \\
&\qquad = \sum_{\xi\in\Xi} \xi \sum_{y\in V_i(\xi)} \sum_{y_i\in V_i(\xi,y)} \frac{1}{|A|} \, \mathbb{P}_w\big(z_r = y_r, \; r \in R_i; \; \tau(R) = i\big) \\
&\qquad = \sum_{\xi\in\Xi} \xi \sum_{y\in V_i(\xi)} \frac{|V_i(\xi, y)|}{|A|} \, \mathbb{P}_w\big(z_r = y_r, \; r \in R_i; \; \tau(R) = i\big) \\
&\qquad = \frac{|Ker(\chi_i)|}{|A|} \sum_{\xi\in\Xi} \xi \sum_{y\in V_i(\xi)} \mathbb{P}_w\big(z_r = y_r, \; r \in R_i; \; \tau(R) = i\big)
\end{aligned}$$

$$=\frac{|Ker(\chi_i)|}{|A|}\sum_{y\in A^{R_i}}\mathbb{P}_w\big(z_r=y_r,\ r\in R_i;\ \tau(R)=i\big)\cdot\sum_{\{\xi\in\Xi:V_i(\xi,y)\neq\emptyset\}}\xi$$

where in the second equality we have used Theorem 5.3. Recall $\xi_y = \chi_{R_i}(y)$. We have

$$\{\xi\in\Xi : V_i(\xi,y)\neq\emptyset\}=\{\xi\in\Xi:\chi_i^{-1}(\xi\xi_y^{-1})\neq\emptyset\}=\xi_y\chi_i(A)\ .$$

Hence

$$\sum_{\{\xi\in\Xi:V_i(\xi,y)\neq\emptyset\}}\xi=\xi_y\sum_{\xi\in\chi_i(K)}\xi=0\ .$$

We conclude

$$\sum_{\xi\in\Xi}\xi\ \mathbb{P}_w\big(\chi(r_R)=\xi,\ \tau(R)=i\big)=0\ .$$

Coming back to the integral we get,

$$\begin{aligned}\Big|\int_{A^{\mathbb{Z}}}\chi(x_R)d\mu_w(\mathbf{x})\Big| &= \Big|\sum_{\xi\in\Xi}\xi\ \mathbb{P}_w\big(\chi(x_R)=\xi,\ \tau(R)=\infty\big)\Big|\\ &\leq |\Xi|\,\mathbb{P}_w\big(\tau(R)=\infty\big)\leq |K|\,\rho(|R|)\ .\end{aligned}$$

Since this inequality holds for any $w\in A^{-\mathbb{N}}$ we have $\left|\int\chi(\mathbf{x})\,d\mu(\mathbf{x})\right|\leq |A|\,\rho(|R|)$. Since $\rho(|R|)\to 0$ as $|R|\to\infty$ we conclude that μ is harmonically mixing. □

5.3. **Some generalizations.** There are several extensions of Theorem 5.3. Just to give the flavour of them here we give one where fullshifts are changed by subgroup shifts. Other generalizations appeared in [16, 24].

Theorem 5.5 (Maass, Martínez, Pivato, Yassawi, [14, 15])**.** *Let $\mathcal{G}\subseteq A^{\mathbb{Z}}$ be an irreducible subgroup shift verifying the following-lifting-property (resp. A is p^s-torsion with p prime). Let $F:\mathcal{G}\to\mathcal{G}$ be a proper linear block map and μ a probability measure with complete connections and summable memory decay compatible with $\mathcal{G}$. Then, the Cesàro mean of μ under the action of F converges to the Haar measure of $\mathcal{G}$. If A is a p-group with p-prime $\mathcal{G}$ always verifies the FLP property.*

6. Final comments and questions

– We would like to change the “complete connections and summable decay of correlations” property by some *mixing* property for the shift map.

– The asymptotic randomization does not require full support of the initial measure and positive entropy w.r.t. the shift map: there exist shift invariant measures μ on $\{0,1\}^{\mathbb{Z}}$ with $h_\mu(\sigma) = 0$ that are asymptotically randomized by $F = id + \sigma$ (see Pivato and Yassawi examples in [22]).

— *Question:* Do Cesàro means exist for expansive and/or positively expansive block-maps of a mixing shift of finite type ? how the limit is related with the unique maximal entropy measure ? There are some partial results for classes of right permutative cellular automata: with associative local rules, or N-scaling local rules (see [10]). These CA can be seen as the product of an algebraic CA with a shift. Here measures are not asymptotically randomized but the limits are the product of a maximal measure with a periodic measure, so combining results from [6, 20, 21] and [2].

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