CHAPTER 1

Dynamical approach of Delone sets and tilings

1. Definitions and backgrounds on Delone sets

In this subsection, we present the basic definitions and properties concerning Delone sets. The Euclidean space of dimension $d \geq 1$ is denoted $\mathbb{R}^d$ and the open ball of radius R centered at $x \in \mathbb{R}^d$ is denoted $B_R(x)$. For any set $X \subset \mathbb{R}^d$ and point $y \in \mathbb{R}^d$, $X - y$ denotes the set $\{y - x; y \in X\}$. More details on Delone sets can be found, for instance, in [**BBG06, KP00, LP03**].

Definition of a Delone set. A discrete subset X of $\mathbb{R}^d$ is a (r_X, R_X)-Delone set if X is:

- r_X-*uniformly discrete*: each open ball of radius r_X in $\mathbb{R}^d$ contains at most one point of X,
- R_X-*relatively dense*: each closed ball of radius R_X in $\mathbb{R}^d$ contains at least one point of X.

If no precision is required, we just say that X is a Delone set. For instance the subset $\mathbb{Z} \times \{0\}$ is not a Delone set of $\mathbb{R}^2$, whereas the lattice of points with integer coordinates $\mathbb{Z}^d$ is a Delone set. Moreover, the image of a Delone set by a bi-Lipschitz map (a bijective Lipschitz map whose inverse also is Lipschitz) of $\mathbb{R}^d$ is still a Delone set.
For $R > R_X$, a subset $\mathtt{P}$ of a Delone set X is a R-*patch* of X, or a *pattern* for short, if, for some $x \in X$, one has

$$\mathtt{P} = X \cap \overline{B_R(x)}.$$

We say that the patch $\mathtt{P}$ is *centered at* x, but noticed that this center may not be unique. A point $y \in \mathbb{R}^d$ is an *occurence* of a patch $\mathtt{P} = X \cap \overline{B_R(x)}$ if

$$(X \cap \overline{B_R(y)}) - y = (X \cap \overline{B_R(x)}) - x.$$

The R-*atlas* $\mathcal{A}_X(R)$ of X is the collection of all the R-patches centered at a point of X and translated to the origin. More precisely, we set

$$\mathcal{A}_X(R) := \{(X \cap \overline{B_R(x)}) - x : \ x \in X\}.$$

We say that X has a *finite local complexity* (FLC) if $\mathcal{A}_X(R)$ is finite for every R. This means that X has a finite number of R-patches, up to translation.

Exercice 1. *Show that a Delone set X has a finite local complexity if and only if the set*

$$X - X := \{x - y; \ x, y \in X\}$$

is discrete, or equivalently, the intersection of any bounded set with $X - X$ is finite.

Exercice 2. *Show that a Delone set X has a finite local complexity iff there exists an $R > 2R_X$ such that $\sharp\mathcal{A}_X(R) < +\infty$.*

Of course, the lattice $\mathbb{Z}^d$ has finite local complexity, since the set $\mathbb{Z}^d - \mathbb{Z}^d = \mathbb{Z}^d$ is uniformly discrete. If α is an irrational number, let $f\colon \mathbb{R} \to \mathbb{R}$ denote the map

$$f\colon y \mapsto \begin{cases} 0 & \text{when } y \text{ is odd,} \\ \alpha\frac{y}{2} & \text{otherwise.} \end{cases}$$

Then, the set $\{(x + f(y), y); x, y \in \mathbb{Z}\}$ has no finite local complexity.
Less trivial examples of FLC Delone sets are given by the *cut and project scheme*. We present here a construction in dimension one that can be straightforwardly generalized in any dimension. Let α be number and let $\mathcal{D}$ be the line $y = \alpha x$ in $\mathbb{R}^2$. We denote by $\pi\colon \mathbb{R}^2 \to \mathcal{D}$ and $\pi^{\perp}\colon \mathbb{R}^2 \to \mathcal{D}^{\perp}$ the orthogonal projections respectively onto $\mathcal{D}$ and its orthogonal space $\mathcal{D}^{\perp}$.

Exercice 3. *Show that the projection on $\mathcal{D}$ of integer points at distance 1 from $\mathcal{D}$, namely*

$$X(\alpha) := \pi\{z \in \mathbb{Z}^2; \|\pi^{\perp}(z)\| \leq 1\}$$

forms a Delone set of $\mathbb{R}$ with finite local complexity.

Observe that, when X is a Delone set of $\mathbb{R}^d$, then $X + v$, obtained by translating any point of X by $v \in \mathbb{R}^d$, also is a Delone set. A Delone set is said to be *aperiodic* if $X + v = X$ implies $v = 0$, and *periodic* otherwise. In Example 3, it is simple to check that the Delone set $X(\alpha)$ is aperiodic if, and only if, α is irrational.
The collection of return vectors associated to the patch $\mathtt{P}$ is thus the set

$$\mathcal{R}_{\mathtt{P}}(X) := \{v \in \mathbb{R} : \mathtt{P} + v \text{ is a patch of } X\}.$$

If we fix a center $x_{\mathtt{P}}$ of $\mathtt{P}$ such that $\mathtt{P} = X \cap \overline{B_R(x_{\mathtt{P}})}$ then the *set of occurrences* of $\mathtt{P}$ is the set

$$X_{\mathtt{P}} := x_{\mathtt{P}} + \mathcal{R}_{\mathtt{P}}.$$

The following lemma is well-known and its proof is plain by contradiction.

Lemma 1 (Repulsion Lemma). *If X is an aperiodic FLC Delone set, then, given $S > 0$, there exists a constant $R_S > 0$ such that, for any $R \geq R_S$ and any R-patch $\mathtt{P}$ of X, its set of occurrences $X_{\mathtt{P}}$ is S-uniformly discrete.*

Exercice 4. *Prove the repulsion lemma.*

There is a relation between Delone set and tiling of the space. For a (r_X, R_X)-Delone set X, the *Voronoï cell* V_x of a point $x \in X$ is the set

$$V_x = \{y \in \mathbb{R}^d : \|y - x\| \leq \|y - x'\|, \forall x' \in X\}.$$

It is then direct to check that any Voronoï cell V_x is a convex polyhedra, its diameter is smaller or equal to $2R_X$ and it contains the ball $B_{\frac{r_X}{2}}(x)$. Moreover when the Delone set X is of finite local complexity, the collection of Voronoï cells $\{V_x\}_{x\in X}$ forms a tiling of $\mathbb{R}^d$, where the tiles meet full face to full face. This last condition implies there is a finite number of pattern (connected set of tiles) for a fixed diameter.
Conversely, given a tiling with a finite number of tiles, up to translation, and assume that each tile is a convex polyhedra and the tiles meet full face to full face. Then by choosing a point on the barycenter of each tile, we get a FLC Delone set. We prefer to use the Delone set point of view because it enables to avoid to manage the geometry of the tiles, that is useless for our purpose.

2. Topology of Delone sets spaces

In order to study relevant combinatorial properties of a Delone set X, we will associate a dynamical system, whose properties reflect the combinatorial one of X. This relation between combinatorics and dynamics is actually classical. For instance, such strategy has been successfully used by Furstenberg to give a proof of the Szemeredi's Theorem [**Fur81**]. For detailed proofs, we refer to [**BBG06, Rob96, Sol97, Rud89**].

First, observe that to each Delone set X, we can associate a Radon measure (taking a finite value to each compact set) of $\mathbb{R}^d$, by the following way $\nu_X = \sum_{x\in X} \delta_x$, where δ_x is the Dirac measure at x. The set of theses measures $\mathcal{M}(\mathbb{R}^d, r, R)$ obtained from (r, R)-Delone sets, is a subset of the dual of continuous functions with compact support $\mathcal{C}_c(\mathbb{R}^d, \mathbb{R})$. It is a closed set for the weak-* topology. The induced topology on this set is called the *Gromov-Hausdorff* topology and is metrizable. For this topology, a sequence $(X_n)_n$ of Delone sets converge if and only if for each bounded open set $U \subset \mathbb{R}^d$, the sequence of sets $(U \cap X_n)_n$ converges for the Hausdorff topology. More combinatorially, for this topology, two Delone sets are close if they agree on a big ball centered at the origin, up to a small translation.

Exercice 5. *Show that the following define a distance for the Gromov-Hausdorff topology on the set of FLC Delone sets.*

$$D(X_1, X_2) := \inf\left(\left\{0 < r < 1 : \exists \|v\| < r,\ s.t.\ (X_1 \cap B_{1/r}(0)) + v = X_2 \cap B_{1/r}(0)\right\} \cup \{1\}\right).$$

The group $\mathbb{R}^d$ acts continuously by translation on the set of FLC Delone sets

$$\begin{array}{rcl} \mathbb{R}^d \times \{X \text{ FLC Delone sets }\} & \to & \{X \text{ FLC Delone sets }\} \\ (v, X) & \mapsto & X - v := \{x - v; x \in X\}. \end{array}$$

This action can also be seen as the restriction of the $\mathbb{R}^d$-translation action on $\mathcal{M}(\mathbb{R}^d, r, R)$ given by $\nu.v(f) := \nu(f(\cdot - v))$ for $\nu \in \mathcal{M}(\mathbb{R}^d, r, R)$, $f \in \mathcal{C}_c(\mathbb{R}^d, \mathbb{R})$ and $v \in \mathbb{R}^d$. The orbit of X is denoted $X + \mathbb{R}^d = \{X + v : v \in \mathbb{R}^d\}$.

The *continuous hull* $\Omega(X)$ of a Delone set X is the closure for the Gromov-Hausdorff topology of the $\mathbb{R}^d$-orbit of X, $\Omega(X) = \overline{X + \mathbb{R}^d}$. So $\Omega(X)$ is a set of Delone sets and is invariant by the translation action. The dynamical system $(\Omega(X), \mathbb{R}^d)$ is then called a *Delone system*. Notice that

$$\forall X' \in \Omega(X), \text{ we have } \mathcal{A}_R(X') \subset \mathcal{A}_R(X) \text{ for every } R > 0. \tag{1}$$

Property 1. *Let $X \subset \mathbb{R}^d$ be a FLC Delone set, then the hull $\Omega(X)$ is compact.*

Proof. Since the topology is metrizable, it is enough to show the compacity with sequences. Let $(X_n)_n \subset \Omega(X)$ be a sequence of Delone sets in the hull. By (1), we get that each X_n is a (r_X, R_X)-Delone set. So there is a $v_n \in \mathbb{R}^d$, $\|v_n\| \leq R_X$ such that the origin 0 belongs to the set $X_n - v_n$. Since X has FLC, from (1), we also get that for any real $R > 0$, the patch $(X_n - v_n) \cap B_R(0)$ is the same for infinitely many indices n. So by a diagonal extraction, we obtain that a subsequence of $(X_n - v_n)_n$ converges to a Delone set $Y \subset \mathbb{R}^d$. By compacity, we may also assume that the sequence $(v_n)_n$ converges to a vectors $v \in \mathbb{R}^d$. It follows that a subsequence of $(X_n)_n$ converges to the Delone set $Y + v$. □

Examples of hull. Let X be Delone set of $\mathbb{R}^d$ with d independent periods $X + v_i = X$ for $v_1, \ldots, v_d$ independent vectors. Then the is an onto continuous map $\mathbb{R}^d/\langle v_1, \ldots, v_d\rangle \to X + \mathbb{R}^d$, so that the hull $\Omega(X)$ is homeomorphic to the d-torus $\mathbb{R}^d/\langle v_1, \ldots, v_d\rangle$.

When the Delone set X is aperiodic, the things are more complicated, but detailed in Section 4.

3. Multirecurrence theorem

We will see in this section an application of Dynamical systems theory to the combinatorics of the Delone sets. For this, let us first recall the Multiple Birkhoff Recurrence Theorem for commuting homeomorphism [**Fur81**, Proposition 2.5].

THEOREM 1 (Multiple Birkhoff Recurrence Theorem (Furstenberg)). *Let Ω be a compact metric space and $T_1, \dots, T_\ell$ commuting homeomorphisms of Ω. Then there exist a point $x \in \Omega$ and a sequence $n_k \to \infty$ such that for each $i = 1, \dots, \ell$, $T_i^{n_k}(x) \to x$ as k goes to infinity.*

The main important fact in this theorem is that the sequence $(n_k)_k$ is the same for each index i. Furstenberg's original application of this theorem was to prove Gallai's extension of the Van der Waerden's theorem to higher dimension.

COROLLARY 1. *Let $X \subset \mathbb{R}^d$ be a Delone set with FLC. Given $\epsilon > 0$ and a finite subset $F \subset \mathbb{R}^d$, there exists an $n \in \mathbb{N}$ and a $1/\epsilon$-patch $\mathtt{P}$ of X such that for each $u \in F$ there exists a vector $\vec{c}$, $\|\vec{c}\| < \epsilon$ such that*

$$\mathtt{P} + nu + \vec{c} \text{ is a patch of } X.$$

This result means that for every FLC Delone set X and every arbitrary finite set F, there exists a big patch of X whose the set of occurrences contains an homothetic copy of F, up to a small error. See [**dlLW09**] for extensions of this results.

PROOF OF COROLLARY 1. We known from Proposition 1 that the hull $\Omega(X)$ is a compact metric space. Let $F = \{u_1, \dots, u_\ell\}$ and consider ℓ commuting homeomorphism of $\Omega(X)$ given by $T_i \colon Y \mapsto Y - u_i$. Consider $\epsilon' > 0$ such that $1/\epsilon > 1/\epsilon + R_X + 1$ where R_X is the relative denseness constant of X.
By the Multiple Birkhoff Recurrence Theorem, there exists a Delone set $Y \in \Omega(X)$ an a sequence $n_k \to \infty$ such that $T_i^{n_k} Y \to Y$ for $1 \leq i \leq \ell$. In particular there is an integer n such that $D(Y, T_i^n Y) < \epsilon'$ for each index i. Since Y is a limit of translated of X, we can find a vector $v \in \mathbb{R}^d$, such that $D(X - v - nu_i, X - v) < \epsilon'$. By definition of the metric, the Delone sets $X - v - nu_i$ and $X - v$ coincide, up to an ϵ' translation, on a ball of radius $1/\epsilon'$ centered at the origin. This means there exist a $1/\epsilon' - R_X$-patch $\mathtt{P}$ of X and vector $\vec{c_i}$, $\|\vec{c_i}\| < \epsilon$ such that $\mathtt{P} + nu_i + \vec{c_i}$ is a patch of $X - v$, and so is a patch of X. $\square$

4. Topological dynamics

We present general properties of dynamical systems. To simplify the presentation, we restrict ourself on a continus $\mathbb{R}^d$-action on a compact metric space Ω, denoted by $(v, x) \in \mathbb{R}^d \times \Omega \mapsto x - v \in \Omega$. But all of this can be generalized to continuous action of, not necessarily Abelian, topological group (see [**Aus88**]).

DEFINITION 1. *Let $(\Omega, \mathbb{R}^d)$ be a topological dynamical system where Ω is a compact metric space. A closed $\mathbb{R}^d$-invariant subset $K \subset \Omega$ is* minimal *for the action if there is no non empty closed subset $K' \subsetneq K$ invariant by the $\mathbb{R}^d$ action.*

Let us precise that K is invariant by the action means when $K - v = K$ for each vector $v \in \mathbb{R}^d$.
For instance, if the orbit of a point is compact, this orbit is a minimal subset.
The following characterization is easy: A set K is minimal for the $\mathbb{R}^d$-action if, and only if, $\overline{orb(x)} = K$ for all $x \in K$. In this sense, it is not possible to topologically separate the orbits in a minimal set.
The interest of this notion comes from the following result:

THEOREM 2. *Let $(\Omega, \mathbb{R}^d)$ be a topological dynamical system where Ω is a compact metric space. Then there exists a non empty compact minimal subset invariant by the $\mathbb{R}^d$ action.*

PROOF. It is enough to apply the Zorn's lemma to the collection of invariant closed and non empty subset of Ω, with the order relation given by the inclusion. This collection is non empty since it contains the set Ω. If we consider a chain, *i.e.* a countable family of nested compact invariant subsets. The intersection is a non empty compact subset and invariant by the action, it is a smallest element. By Zorn's lemma, there exists a smallest non empty compact subset invariant by the action. $\square$

THEOREM 3 (Auslander, [**Aus88**]). *Let $(\Omega, \mathbb{R}^d)$ be a topological dynamical system where Ω is a compact metric space and $x \in \Omega$. The closure of the $\mathbb{R}^d$-orbit of x is minimal $\Leftrightarrow$ the point x is* almost periodic*,* i.e. *for any neighborhood U of x, the set $\{v \in \mathbb{R}^d; x - v \in U\}$ is relatively dense.*

PROOF. $\Rightarrow$ Assume that $M := \overline{orb(x)}$ is minimal and let U be a neighborhood of x. First note that $M \subset U - \mathbb{R}^d$ (otherwise $M \setminus (U - \mathbb{R}^d)$ is a closed invariant proper subset of M). By compactness, there exists a finite set $K = \{v_1, \ldots, v_\ell\} \subset \mathbb{R}^d$ such that $M \subset \cup_{i=1}^{\ell} U - v_i$. It follows that for any $w \in \mathbb{R}^d$, there is a v_i such that $x - (w - v_i) \in U$. In other terms, we have $\mathbb{R}^d = \{v \in \mathbb{R}^d; x - v \in U\} + K$, and we get the conclusion.
$\Leftarrow$ Suppose that x is an almost periodic point. If the compact set $M := \overline{orb(x)}$ is not minimal, by lemma 2, there exists a non empty minimal set $M' \subsetneq M$. Let U and V be disjoint open sets with $x \in U$ and $M' \subset V$. Let $K \subset \mathbb{R}^d$ be an arbitrary compact set and let W be a neighborhood of M' such that $W \subset V - K$. Since the orbit of x is dense, there is a $w \in \mathbb{R}^d$ such that $x - w \in W$. Hence $x - w + K \subset W + K \subset V$ and we get that $(x - w + K) \cap U = \emptyset$. We have then $\{v \in \mathbb{R}^d; x - v \in U\} + K \neq \mathbb{R}^d$. Since K is arbitrary, we obtain a contradiction with the almost periodicity of x. $\square$

The almost periodicity can be interpreted combinatorially for Delone sets.

COROLLARY 2. *Let $X \subset \mathbb{R}^d$ be a FLC Delone set. The hull $\Omega(X)$ is minimal if, and only if, the set X is* repetitive*:* i.e. *For any real $R > 0$, there is a constant $M(R) > 0$ such that any ball $B_M(x)$ of radius M, the set $B_M(x) \cap X$ contains an occurrence of each R-patch of X.*

PROOF. Exercice ! $\square$

From this and the repulsion lemma (Lemma 1), we get:

COROLLARY 3. *Let $X \subset \mathbb{R}^d$ be a repetitive aperiodic FLC Delone set.*
Then for any patch $\mathtt{P}$, its set of occurrences $X_{\mathtt{P}}$ is a FLC Delone set.
Each Delone set $Y \in \Omega(X)$ is a repetitive aperiodic Delone set.

EXERCICE 6. *Let α be an irrational number. Show that the set $\mathbb{Z}\alpha + \mathbb{Z} = \{n\alpha + m; n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Deduce from this that the Delone set $X(\alpha)$ defined in Exercice 3 is repetitive.*

5. Geometrical properties of the hull

Let $X \subset \mathbb{R}^d$ be a FLC Delone set.
Given $Y \in \Omega(X)$ and $S > 0$ such that $Y \cap B_S(0) \neq \emptyset$, the associated *cylinder set* is defined as

$$C_{Y,S} := \{Y' \in \Omega(X) : Y \cap \overline{B_S(0)} = Y' \cap \overline{B_S(0)}\}.$$

The translations of cylinder sets, namely,

$$C_{Y,S} - B_\epsilon(0) := \{Y' - v : v \in B_\epsilon(0),\ Y' \in C_{Y,S}\}, \qquad \text{for } \epsilon > 0,\ S > 0,\ Y \in \Omega(X),$$

form a base for the topology of $\Omega(X)$.
The *canonical transversal* $\Xi_0(X)$ of the hull $\Omega(X)$ is the set of Delone sets Y in $\Omega(X)$ such that the origin 0 belongs to Y. The designation transversal comes from the obvious fact that the set $\Xi_0(X)$ is transverse to the action: for any vector v smaller than the uniform discreteness constant, clearly $Y - v \notin \Xi_0(X)$ for any $Y \in \Xi_0(X)$. This gives a Poincaré section.

Property 2 ([**KP00**]). *The canonical transversal $\Xi_0(X)$ and the cylinder sets $C_{X,S}$ of a repetitive aperiodic FLC Delone set $X \subset \mathbb{R}^d$ are Cantor sets.*

Observe that when X has d independent periods $v_1, \dots, v_d \in \mathbb{R}^d$, $X - v_i = X$, the canonical transversal and the cylinder sets are finite.

Proof. Observe that the canonical transversal and each cylinder set are compact sets for the Gromov-Hausdorff topology. Moreover, the FLC property implies that for every $S > 0$, $Y \in \Xi_0(X)$, the complementary in $\Xi_0(X)$, of the cylinder set $C_{Y,S}$, is a finite union of cylinder sets, that is $\Xi_0(X) \setminus C_{Y,S} = \bigcup_{i=1}^{\ell} C_{Y_i,S}$ for some $Y_1, \dots, Y_\ell$ in $\Xi_0(X)$. It follows that $\Xi_0(X) \setminus C_{Y,S}$ is a closed set, meaning that $C_{Y,S}$ is both closed and open. Since cylinder sets generate the topology on $\Xi_0(X)$, this space is totally disconnected.
If $\Xi_0(X)$ has an isolated point Y, the repetitivity hypothesis, implies there are infinitely many vectors $v \in \mathbb{R}^d$ such that $X - v = Y$. This is a contradiction with the aperiodicity of X. Finally we get that $\Xi_0(X)$ is a Cantor set. $\square$

Property 3. *Let $X \subset \mathbb{R}^d$ be repetitive aperiodic FLC Delone set. Then for any $S > 0$ there exists a real R, such that the map*

$$\begin{array}{rcl} B_S(0) \times C_{X,R} & \to & C_{X,R} - B_S(0) \subset \Omega(X) \\ (v, Y) & \mapsto & Y - v \end{array}$$

is a homeomorphism onto its image.

As a consequence, the hull is locally homeomorphic to the Cartesian product of an open subset of $\mathbb{R}^d$ times a Cantor set. This is a structure of a *matchbox manifold* [**AO95**]. Each orbit corresponds then to a pathwise connected component of the topological space. Geometrically, these subsets are called *leaves*. In the aperiodic case, each leaf (or each orbit) is homeormorphic to $\mathbb{R}^d$. Moreover the transversal space (to he leaves) is totally disconnected.
More specifically, one can show that the hull has a *flat-lamination structure* meaning that the transition map along the leaves are translations, and small enough cylinder set share the same vector of translation. This enable to give an other description of the hull in terms of inverse limit of branched manifolds [**BBG06**]. This construction leads to the proof of the gap-labelling conjecture.

PROOF. The repulsion lemma (Lemma 1) give us a real R such that the set $\{v \in \mathbb{R}^d; X-v \in C_{X,R}\}$ is $2S$-uniformly discrete. Since any patch of $Y \in \Omega(X)$ is still a patch of X, up to translations, the set $\{v \in \mathbb{R}^d; Y - v \in C_{X,R}\}$ is still $2S$-uniformly discrete. The continuity of the map comes from the continuity of the action. To show the injectivity, if $Y_1 - v_1 = Y_2 - v_2$, we have $Y_1 = Y_2 - (v_2 - v_1)$, and so $v_1 = v_2$ by the very choice of the constants. □

EXERCICE 7. The aim of this exercice is to show that for each aperiodic FLC Delone set $X \subset \mathbb{R}^d$, there exist two Delone sets $X_1 \neq X_2 \in \Omega(X)$ and a vector $\vec{v} \in \mathbb{R}^d$, $\|\vec{v}\| = 1$ such that

$$X_1 \cap \{x \in \mathbb{R}^d; \langle x, \vec{v}\rangle > 0\} = X_2 \cap \{x \in \mathbb{R}^d; \langle x, \vec{v}\rangle > 0\},$$

where $\langle \cdot, \cdot \rangle$ denote the usual inner product.
In the following, $\mathbb{S}^{d-1}$ denote the set of vector $\vec{v} \in \mathbb{R}^d$ such that $\|\vec{v}\| = 1$ and $H_{\vec{v}}$ is the open half-space $\{x \in \mathbb{R}^d; \langle x, \vec{v}\rangle < 0\}$. For $\epsilon > 0$, let

$$E_\epsilon := \{R > 0; \exists \vec{v} \in \mathbb{S}^{d-1}, Y, Z \in \Omega(X), D(X, Y) \geq \epsilon, \sup_{w \in H_{\vec{v}} \cap B_R(0)} D(Y - w, Z - w) < \epsilon\}.$$

Let $M_\epsilon = \sup E_\epsilon$ if $E_\epsilon \neq \emptyset$ and $M_\epsilon = 1$ otherwise.
0) Show that if $M_\epsilon = +\infty$ for an $\epsilon > 0$ small enough, then we get the conclusion.
By contradiction. We assume now that $M_\epsilon < +\infty$ for every $\epsilon > 0$.
1) Show there exists a $R_0 > 0$ such that for every vector $w \in \mathbb{R}^d$, $\|w\| \geq R_0$, there is an half space $H_{\vec{v}}$ such that

$$H_{\vec{v}} \cap B_{M_\epsilon+1}(0) \subset B_{\|w\|}(w).$$

2) Let $R_1 = \max(R_0, M_\epsilon + 1)$. Show there exists a $\delta > 0$ such that if $D(Y, Z) < \delta$, $Y, Z \in \Omega(X)$ then $\sup_{w \in B_{R_1}(0)} D(Y - w, Z - w) < \epsilon$.
3) Let $Y, Z \in \Omega(X)$ such that $D(Y, Z) < \delta$. Show that the set $\{R, D(Y - w, Z - w) < \epsilon, \forall w \in B_R(0)\}$ is an open interval of $\mathbb{R}$.
Let $R_\infty := \sup\{R, D(Y - w, Z - w) < \epsilon, \forall w \in B_R(0)\}$. Show that $R_\infty \geq R_1$.
Let us assume that $R_\infty < +\infty$. Let $w_0 \in \mathbb{R}^d$ such that $\|w_0\| = R_\infty$. Prove that for any $w \in H_{\vec{v}} \cap B_{M_\epsilon+1}(0)$, we have $w + w_0 \in B_{\|w_0\|}(0)$.
Deduce that $D(Y - w_0, Z - w_0) < \epsilon$, and obtain a contradiction to obtain that $R_\infty = +\infty$.
4) Deduce from 3) that X is periodic.

CHAPTER 2

Linearly repetititve Delone sets

1. History and motivations

The notion of *linearly recurrent subshift* has been introduced in [**DHS99**] to study the relations between substitutive dynamical systems and stationary dimension groups. In an independent way, the similar notion of *linearly repetitive Delone sets* of the Euclidean space $\mathbb{R}^d$ appears in [**LP02**]. For a Delone set X of $\mathbb{R}^d$ the *repetitivity function* $M_X(R)$ is the least M (possibly infinite) such that every closed ball B of radius M intersected with X contains a translated copy of any R-patch. Recall from Corollary 2 a finite repetititve function is equivalent to the minimality of the associated hull.
A Delone set X is said *linearly repetitive* if there exists a constant L such that $M_X(R) < LR$ for all $R > 0$. Observe that we can assume that the constant L is greater than 1. According to the following theorem, the slowest growth for the repetitivity function of an aperiodic Delone set is linear.

THEOREM 4 ([**LP02**] Theo. 2.3). *Let $d \geq 1$. There exists a constant $c(d) > 0$ such that for any Delone set X of $\mathbb{R}^d$ such that*

$$M_X(R) < c(d)R \qquad \textit{for some } R > 0,$$

then X has a non-zero period.

Even more, if for some R, $M_X(R) < \frac{4}{3}R$, then the Delone set X is a *crystal i.e.* has d independent periods (Theo. 2.2 [**LP02**]) .
The classical examples of aperiodic Delone systems, e.g. the ones arising from substitutions, are linearly repetitive.

LEMMA 2 ([**Sol98**] Lem. 2.3). *A primitive self similar tiling is linearly repetitive.*

In many senses that we will not specify, the family of linearly repetitive Delone sets is small inside the family of all the Delone sets of the Euclidean space $\mathbb{R}^d$. For instance, in the class of Sturmian subshifts, several authors [**MH40, Dur00, LP03**] show the following result.

PROPERTY 4. *The Sturmian subshift associated to an irrational number α is linearly recurrent if and only if the coefficients of the continued fraction of α are bounded.*

Let us recall that for the standard topology, the set of numbers with bounded continued fraction are badly approximable by rational numbers. It is known that they form a Baire meager set, with 0 Lebesgue measure but with Hausdorff dimension 1.
As we shall see, the linearly repetitive Delone sets possess many rigid properties. In the next section we present some combinatorial properties of these sets. For instance, their complexity appears to be the slowest possible among all the aperiodic repetitive Delone sets. We focus in Section 3 on the ergodic properties of dynamical systems associated to linearly repetitive Delone sets. They are

strictly ergodic (i.e. each patch appears with a frequency). But they are not wild since they are never measurably mixing. We present a characterization of the linear repetitivity by using a bound on the frequencies of the occurrences of the patches. The dynamical factors of these systems are studied in Section 4. They admit as factors just a finite number of non conjugate aperiodic Delone systems. The last section concerns the deformation of linearly repetitive Delone sets: each one is the image through a Lipschitz map of a lattice in $\mathbb{R}^d$. We refer to [**APCC$^+$15**] and its references for a survey on linearly repetitive Delone sets.

2. Combinatorial properties

In this section we give the basic definitions and combinatorial properties concerning linearly repetitive Delone sets of $\mathbb{R}^d$. Most of these properties are obvious for self-similar tilings.
The following lemma shows that two occurrences of a patch can not be too close. The proof can be found in [**Len04**] Lem. 2.1 and in [**Sol98, Dur00**].

Lemma 3. *Let X be a linearly repetitive aperiodic Delone set with constant $L > 1$. Then, for every patch $\mathtt{P} = X \cap B_R(x)$ with $x \in X$, $R > 0$, its set of occurences $X_{\mathtt{P}}$ is a (r_X, R_X)-Delone set where*

$$\frac{R}{L+1} \leq r_{X_{\mathtt{P}}} \leq R_{X_{\mathtt{P}}} \leq LR.$$

Proof. Only the left inequality is not obvious. By contradiction: let us assume there exist $x \neq y \in X$ with

$$(X \cap B_R(x)) - x = X \cap B_R(y) - y$$

and

$$r_X \leq \|x - y\| < \frac{R}{(L+1)}.$$

Then for any point z' in $B_R(x) \cap X$, we have $z' + (y - x) \in X$. For any $z \in X$, the set $X \cap B_R(x)$ contains a translated copy centered in $z' \in X \cap B_R(x)$ of the patch $B_{\frac{R}{L+1}}(z) \cap X$. Thus $z' + (y-x) \in X \cap B_{\frac{R}{L+1}}(z')$ and finally $z + (y - x) \in X$ and so $X + (y - x) \subset X$. In a similar way we obtain $X + (x - y) \subset X$, so that finally we get $X + x - y = X$ contradicting the aperiodicity of X. $\square$

This repulsion property on the occurrences of patches has several consequences on the combinatorics of the Delone set X.
First of all on the complexity. Let us denote $N_X(R)$ the number of different R-patches $B_R(x) \cap X$ with $x \in X$, up to translation. Since any ball of radius $M_X(R)$ contains the occurrences of any R-patch, we easily deduce that $N_X(R)^{\frac{1}{d}} = O(M_X(R))$ as $R \to \infty$ (see [**LP03**]).

Lemma 4 ([**Len04**], Lem. 2.2). *Let X be an aperiodic linearly repetitive Delone set. Then*

$$\liminf_{R \to +\infty} \frac{N_X(R)}{R^d} > 0.$$

From this, we conclude that for an aperiodic linearly repetitive Delone set $M_X(R) = O(N_X(R)^{\frac{1}{d}})$ as $R \to \infty$.

Proof. As X is relatively dense, there exist constants $\lambda_1 > 0$ and $R_1 > 0$ such that

$$\sharp(X \cap B_R(x)) \geq \lambda_1 R^d \qquad \text{for any } x \in X,\ R \geq R_1.$$

By the previous lemma all the patches $(X-x)\cap B_R(0)$ for $x \in X \cap B_{\frac{R}{3(L+1)}}(0)$ are pairwise different. Thus for any $R \geq 3(L+1)R_1$, we have

$$N_X(R) \geq \sharp(X \cap B_{\frac{R}{3(L+1)}}(0)) \geq \lambda_1 \left(\frac{R}{3(L+1)}\right)^d,$$

that gives us the result. □

Another property is on the hierarchical structure of the linearly repetitive Delone sets, that is quite simple: for any size $R > 0$, it is possible to decompose the Delone set into big patches (each one containing a R-patch), so that the number of these patches, up to translations, is independent of the size R. To be more precise, we need the notion of *Voronoï cell of a patch*.
For a patch R-patch $\mathtt{P} = B_R(x_{\mathtt{P}}) \cap X$ of a repetitive Delone set X, we denote by $V_{\mathtt{P},x}$ the Voronoï cell (see chapter 1) associated to the Delone set $X_{\mathtt{P}}$ and an occurrence $x \in X_{\mathtt{P}}$.
It follows by Lemma 3 that for an aperiodic linearly repetitive Delone set with constant L, for any R-patch $\mathtt{P}$,

$$\text{diam } V_{\mathtt{P},x} \leq 2LR, \qquad B_{\frac{R}{2(L+1)}}(x) \subset V_{\mathtt{P},x}, \qquad \text{for any } x \in X_{\mathtt{P}}. \tag{2}$$

LEMMA 5 ([**CDP10**] Lem. 11). *Let X be an aperiodic linearly repetitive Delone set with constant L. There exists an explicit positive constant $c(L)$ such that for every $R > 0$ and every R-patch $\mathtt{P} = X \cap B_R(x)$, the collection $\{X \cap V_{\mathtt{P},x} : x \in X_{\mathtt{P}}\}$ contains at most $c(L)$ elements up to translation.*

Observe here that the bound, explicit in the proof, does not depend on the combinatorics of X but just on the constant of repetitivity.

PROOF. Let us consider B the union of Voronoï cells $V_{\mathtt{P},x}$, $x \in X_{\mathtt{P}}$ that intersects the ball $B_{L^2R}(0)$. We have then

$$B_{L^2R}(0) \subset B \subset B_{L^2R+2LR}(0).$$

By linear repetitivity, $B \cap X$ contains a translated copy of any patch of the kind $X \cap V_{\mathtt{P},x}$ with $x \in X_{\mathtt{P}}$. Since any Voronoï cell contains a ball of radius $\frac{R}{2(L+1)}$, the number of patches in $B \cap X$ of the kind $X \cap V_{\mathtt{P},x}$ with $x \in X_{\mathtt{P}}$ is smaller than

$$\frac{\text{vol } B_{RL(L+2)}(0)}{\text{vol } B_{\frac{R}{2(L+1)}}(0)} \leq (2L(L+2)^2)^d = c(L).$$

□

3. Ergodic properties of linearly repetitive system

3.1. Background on invariant measure. As for $\mathbb{Z}$-action, it is possible to show that any contiunous $\mathbb{R}^d$-actions on a compact metric space Ω does admit a *translation invariant* measure μ: that is, a Borel probability measure such that $\mu(B - v) = \mu(B)$ for every Borel set B and $v \in \mathbb{R}^d$. The proof is similar to the one of the Krylov-Bogolyubov theorem, by fixing a point $w_0 \in \Omega$ and a weak-limit of the linear forms $f \in C(\Omega) \mapsto \frac{1}{\text{vol} D_N}\int_{D_N} f(w_0 - t)dLeb(t)$, where $(D_N)_N$ is a nested sequence of d-cubes, and Leb denotes the Lebesgue measure on $\mathbb{R}^d$.
We present here a local description of a probability measure μ on the hull $\Omega(X)$ of a FLC Delone set invariant by the $\mathbb{R}^d$-action. Let C be a cylinder set. Each translation invariant measure μ induces

a measure ν on C (see [**CGSY99**] for the general construction): given a Borel subset V of C, its *transverse measure* is defined by

$$\nu(V) = \frac{\mu(V - B_r(0))}{\mathrm{vol}(B_r(0))},$$

where vol denotes the Euclidean volume in $\mathbb{R}^d$ and with the notations of section 5 in chapter 1. Since the measure μ is $\mathbb{R}^d$-invariant, the value $\nu(V)$ does not depend on small r. This gives a measure on each C. The collection of all measures defined in this way is called the *transverse invariant measure* induced by μ. It is invariant in the sense that if V is a Borel subset of C and $x \in \mathbb{R}^d$ is such that $V - x$ is a Borel subset of another local transversal C', then $\nu(V - x) = \nu(V)$. Conversely, the measure μ of any set written as $C - B_S(0)$, with S small eough, may be computed by the equation

$$\mu(C - B_S(0)) = \mathrm{vol}(B_S(0)) \times \nu(C).$$

3.2. Unique ergodicity and speed of convergence. When the system $(\Omega, \mathbb{R}^d)$ has an unique translation invariant probability measure, the system is called *uniquely ergodic*. The unique ergodicity implies combinatorial properties for the Delone set. The dynamical system $(\Omega, \mathbb{R}^d)$ is uniquely ergodic, if and only if any Delone set $X \in \Omega$ has *uniform patch frequencies*, i.e., any patch $\mathtt{P}$ occurs with a positive frequency; more precisely: Let $X_{\mathtt{P}}$ be the set of occurrences of the patch $\mathtt{P}$ in X, and let $(D_N)_N$ be a nested sequence of d-cube D_N of side N, then the following limit exists.

$$\lim_{N\to\infty} \frac{\sharp X_{\mathtt{P}} \cap D_N}{\mathrm{vol}(D_N)} =: \mathrm{freq}(\mathtt{P}).$$

The number $\mathrm{freq}(\mathtt{P})$ is called the *frequency* of $\mathtt{P}$. Notice the difference with the standard Birkhoff's ergodic Theorem that asserts a convergence only for almost all Delone set of the hull.

THEOREM 5. *Let X be an aperiodic linearly repetitive Delone set of $\mathbb{R}^d$ and Ω its hull. Then the system $(\Omega, \mathbb{R}^d)$ is uniquely ergodic.*

The original proof is due to Lagarias and Pleasants in [**LP03**]. Actually for linearly repetitive system, we can be much more precise and give informations on the speed of convergence of the limit. For instance the following is a stronger result of Lagarias and Pleasants [**LP03**], that implies the unique ergodicity.

THEOREM 6 ([**LP03**]). *Let X be a linearly repetitive Delone set of $\mathbb{R}^d$. There exists a $\delta > 0$ such that, for every patch P of X, there is a number freq(P) so that*

$$\left| \frac{X_P \cap Dom_N}{vol(Dom_N)} - \mathrm{freq}(P) \right| = O(N^{-\delta}),$$

where Dom_N is either a d-cube with side N or a ball of radius N. The O-constant may depend on the patch P.

OPEN QUESTION 1. *Given an $\alpha \in \mathbb{R}$, prove te existence of a FLC Delone set $X \subset \mathbb{R}^d$ that has uniform patch frequencies such that for all patch P of X, $\mathrm{freq}(P) \in \mathbb{Z} + \alpha\mathbb{Z}$.*

3.3. Non-mixing properties. A translation invariant probability measure μ on a the hull Ω of a Delone set is said to be *measurably strongly mixing* if for any Borel sets A, B in Ω,

$$\lim_{\|v\|\to\infty} \mu((A - v) \cap B) = \mu(A)\mu(B). \tag{3}$$

The next result is analogous to a theorem of Dekking and Keane [**DK78**] for substitutive subshifts.

PROPERTY 5 ([**APCC$^+$15**]). *Let X be a linearly repetitive Delone set of $\mathbb{R}^d$ and Ω its hull. Then the system $(\Omega, \mathbb{R}^d)$ is not measurably strongly mixing.*

3.4. A characterization of linear repetitivity. In [**Len02**], D. Lenz characterizes the subshifts that admit a uniform subadditive ergodic Theorem by uniform positivity of weights. This can be considered as an averaged version of linear repetitivity. For Delone systems, it is shown in [**BBL13**] that the linear repetitivity is equivalent to positivity of weights plus some balancedness of the shape of patterns. For a Voronoï cell V of a Delone set, let us define:

$$r_{int} := \sup\{r > 0; V \text{ contains a ball of radius } r\}.$$
$$R_{ext} := \inf\{R > 0; V \text{ is contained in a ball of radius } R\}.$$

The *distorsion* of V is the constant $\lambda(V) := R_{ext}(V)/r_{int}(V)$.

THEOREM 7 ([**BBL13**]). *Let X be an aperiodic Delone set in $\mathbb{R}^d$ of finite type. Then X is linearly repetitive if and only if for any R-patch $\mathtt{P}$ of X, $R > 0$: the set $X_{\mathtt{P}}$ of occurrences of $\mathtt{P}$ is a $(r_{\mathtt{P}}, R_{\mathtt{P}})$-Delone set such that*

(i) $\sup_{\mathtt{P}, x \in X_{\mathtt{P}}} \lambda(V_x) < +\infty$ *where V_x denotes the Voronoï cell of x.*
(ii) *The Delone set X satisfies the positivity of weights :*

$$\inf_{\mathtt{P} \text{ is a } R-patch\ , R \geq R_X} \liminf_{n \to \infty} \frac{\sharp B_n(0) \cap X_{\mathtt{P}}}{vol(B_n(0))} vol(B_R(0)) > 0.$$

One can find in [**BBL13**] another similar equivalent condition to linear repetitivity. Notice that in dimension $d = 1$, the distorsion of any compact Voronoï cell is equal to 1. Thus the condition (ii) is equivalent to the linear repetitivity.

EXERCICE 8. Show the that a LR Delone set satisfies the conditions (i) and (ii) of Theorem 7

EXERCICE 9. This exercice is devoted to show the conditions are suffisant in Theorem 7. So we consider that X is an FLC Delone set satisfying the conditions (i) and (ii).
2) Show that, if there exists a constant $L > 1$ such that for any R-patch $\mathtt{P}$ of X, its set of occurrences is LR-uniformly discrete, then X is linearly repetitive.
3) Assume that X is aperiodic.

a) Show that for R big enough, for any R-patch $\mathtt{P} = R \cap B_R(x)$ of X, if d denote the constant of uniform discreteness of the set of occurrences $X_{\mathtt{P}}$, then $d \geq 3R$. In all the following, we will assume that R would be big enough so that the sentences have a sens. Let m be the maximal integer such that $R + m \leq d/3$. Let $\mathtt{P}_0, \dots, \mathtt{P}_m$ be the patches $\mathtt{P}_i = B_{R+i}(x) \cap X$.

b) Show that the sets of occurrences satisfy $X_{\mathtt{P}_m} \subset \cdots \subset X_{\mathtt{P}_1} \subset X_{\mathtt{P}_0} = X_{\mathtt{P}}$.

c) Prove that for every $i \in \{0, \dots, m\}$, for each $z \in X_{\mathtt{P}_i}$, we have $X_{\mathtt{P}} \cap B_{2(R+i)}(z) = \{z\}$.

d) Deduce that for every $i, j \in \{1, \dots, m\}$, $i \neq j$, we have for every $z \in X_{\mathtt{P}_j}$, $y \in X_{\mathtt{P}_i}$

$$(B_{R+j}(z) \setminus B_{R+j-1}(z)) \cap (B_{R+i}(y) \setminus B_{R+i-1}(y)) = \emptyset$$

e) Deduce from this that for every integer $n \geq 0$,

$$\text{vol}(B_{n+R+m}(0)) \geq \sum_{i=0}^{m} \sharp(B_n(0) \cap X_{\mathtt{P}_i}) \text{vol}(B_{R+i}(0) \setminus B_{R+i-1}(0)).$$

f) Deduce from this and condition (ii) that d/R is bounded uniformly in R.

g) Show that a Delone set satisfying (i) and (ii) is linearly repetitive.

4. Factors of linearly repetitive system

A *factor map* between two Delone systems $(\Omega_1, \mathbb{R}^d)$ and $(\Omega_2, \mathbb{R}^d)$ is a continuous surjective map $\pi : \Omega_1 \to \Omega_2$ such that $\pi(X - v) = \pi(X) - v$, for every $X \in \Omega_1$ and $v \in \mathbb{R}^d$.
In symbolic dynamics it is well-known that topological factor maps between subshifts are always given by sliding-block-codes (Curtis-Hedlund-Lyndon theorem). An equivalent notion for the Delone system is the *pattern equivariant function* $\pi\colon \Omega_1 \to \Omega_2$: *i.e.* there exists a constant $s_0 > 0$, s.t. if two Delone sets $X, Y \in \Omega_1$ satisfy $X \cap B_{s_0}(0) = Y \cap B_{s_0}(0)$ then $\pi(X) \cap \{0\} = \pi(Y) \cap \{0\}$. The scalar s_0 is called the *range* of π.

Exercice 10. *1) Show that for every $R \geq 0$, $X, Y \in \Omega_1$,*

$$X \cap B_{R+s_0}(0) = Y \cap B_{R+s_0}(0) \Rightarrow \pi(X) \cap B_R(0) = \pi(Y) \cap B_R(0).$$

*2) Show that a pattern equivariant function π is continuous and commutes with the action (*i.e. *$\pi(Y - v) = \pi(Y) - v$ for all $v \in \mathbb{R}^d$, $Y \in \Omega_1$).*
3) Prove that the inverse of a bijective pattern equivariant function is still pattern equivariant.

However there are examples of factor maps on Delone systems that are not pattern equivariant functions ([**Pet99**], [**RS01**]). Nevertheless, factor maps between Delone systems are not far from being pattern equivariant [**CD08, CDP10**].

Lemma 6 ([**CD08**] Lem. 3). *Let X_1 and X_2 be two Delone sets with finite local complexity. If $\pi : \Omega(X_1) \to \Omega(X_2)$ is a factor map and X_1 is linearly repetitive, then X_2 is linearly repetitive.*

Exercice 11. *Prove this lemma for a factor map that is a pattern equivariant function.*

The next result says that factor maps between linearly repetitive Delone systems are finite-to-one. A proof of that result in the context of subshifts and Delone systems can be found in [**Dur00**] and in [**CDP10**, Proposition 5] respectively.

Property 6. *Let X be a linearly repetitive Delone set with constant L. There exists a constant $C > 0$ (depending only on L) such that If X' is an aperiodic Delone set and $\pi : (\Omega_X, \mathbb{R}^d) \to (\Omega_{X'}, \mathbb{R}^d)$ is a factor map, then for every $Y \in \Omega_{X'}$, the fiber $\pi^{-1}(\{Y\})$ contains at most C elements.*

Moreover, the number of aperiodic Delone systems that are factors of at the system associated to a linearly repetitive Delone set is finite.

Theorem 8 ([**CDP10**] Theo. 12). *Let $L > 1, d \geq 1$. There exists a constant $N(L, d)$ such that any linearly repetitive Delone set X of $\mathbb{R}^d$ with constant L, has at most $N(L, d)$ non conjugate aperiodic Delone system factors of $(\Omega_X, \mathbb{R}^d)$.*

Exercice 12. *Let X be an aperiodic LR Delone set. Let $Aut_{s_0}(\Omega(X))$ denote the collection of bijective pattern equivalent maps from the hull $\Omega(X)$ to itself, with an associated range smaller than $s_0 > 0$.*
1) Show that for every $s_0 > 0$, $Aut_{s_0}(\Omega(X))$ is finite, and give a bound of its cardinality.
(hint: use Exercice 10)
2) Prove that the growth rate of the group geneerated by a finite set of bijective pattern equivariant functions is at most polynomial.
The growth rate of a group generated by a set S is given for $n \in \mathbb{N}$ by

$$G(n) = \sharp\{g_1 \ldots g_n; g_i \in S \cup S^{-1} \cup \{Id\}\}.$$

5. Bi-Lipschitz equivalence to a lattice

Let X_1 and X_2 be two Delone sets in $\mathbb{R}^d$. We say that they are *bi-Lipschitz equivalent* if there exists a homeomorphism $\phi : X_1 \to X_2$ and a constant $\Delta \geq 1$ such that $\forall x, x' \in X, x \neq x'$

$$\frac{1}{\Delta} \leq \frac{\|\phi(x) - \phi(x')\|}{\|x - x'\|} \leq \Delta.$$

The map ϕ is then called a *bi-Lipschitz homeomorphism* between X_1 and X_2.
The problem to know whether two Delone sets are bi-Lipschitz equivalent was raised by Gromov in [**Gro93**]: for the 2-dimensional Euclidean space: *Is every Delone set in $\mathbb{R}^2$ bi-Lipschitz equivalent to $\mathbb{Z}^2$?* Counterexamples to this question were given independently by Burago and Kleiner [**BK98**] and McMullen [**McM98**]. Moreover, McMullen also showed that when relaxing the bi-Lipschitz condition to a Hölder one, all Delone set (with or without finite local complexity) in $\mathbb{R}^d$ are equivalent. Later, Burago and Kleiner [**BK02**] gave a sufficient condition for a Delone set to be bi-Lipschitz equivalent to $\mathbb{Z}^2$ and asked the following question: *If one forms a Delone set in the plane by placing a point in the center of each tile of a Penrose tiling, is the resulting set bi-Lipschitz equivalent to $\mathbb{Z}^2$?* They studied the more general question of knowing whether a Delone set arising from a cut-and-project tiling is bi-Lipschitz equivalent to $\mathbb{Z}^2$ (recall that the Penrose tiling is also a cut-and project tiling [**dB81**]) and solved it in some cases that do not include the case of Penrose tilings, thus leaving the former question open. Solomon [**Sol11**] gave a positive answer for Penrose tiling by using the fact that it can be constructed using substitutions. In fact, Solomon proved that each Delone set arising from a primitive self-similar tiling in $\mathbb{R}^2$ is bi-Lipschitz to $\mathbb{Z}^2$.
The following result was proved in [**APCG13**].

THEOREM 9. *Every linearly repetitive Delone set in $\mathbb{R}^d$ is bi-Lipschitz equivalent to $\mathbb{Z}^d$.*

Notice that Theorem 9 is trivial when the dimension $d = 1$ since, in this case, every Delone set (with no extra assumptions) is bi-Lipschitz equivalent to $\mathbb{Z}$. Solomon in [**Sol14**] also showed that for every self-similar tiling of $\mathbb{R}^d$ of Pisot type there is a *bounded displacement* between its associated Delone set X and $\beta\mathbb{Z}^d$ for a $\beta > 0$ (i.e. there is a bijection $\phi\colon X \to \beta\mathbb{Z}^d$ such that $\Phi - Id$ is bounded).
The strategy of the proof of Theorem 9 is the following. First consider the easy case where all the Voronoï cells V of a Delone set X have an unit volume. Thus any finite union of N Voronoï cells meet at least N unit squares, and conversely N unit squares meet at least N Voronoï cells. So by the transfinite form of Hall's marriage Lemma, there exists a bijection between the collection of Vornoï cells and the units squares, so that any cell intersects its image. This define a map $\phi\colon X \to \mathbb{Z}^d$ such that $\phi - Id$ is bounded.
For the general case, we need to consider the measurable function $f\colon \mathbb{R}^d \to \mathbb{R}$ defined by

$$f(x) = \sum_{y: x \in V_y} \frac{1}{\mathrm{vol}\ V_y} \qquad x \in \mathbb{R}^d,$$

where V_y denotes the Voronoï cell of the point $y \in X$. If $\phi : \mathbb{R}^d \to \mathbb{R}^d$ is a bi-Lipschitz map so that its Jacobian determinant is f, standard calculus show us that the image $\phi(V)$ of any Voronoï cell V of X has volume 1. The proof of Theorem 9 consists then to generalize to all dimension d a sufficient condition given by Burago and Kleiner [**BK02**] in dimension 2 to solve the equation $\det \mathrm{D}\phi = f$ with ϕ an unknown bi-Lipschitz map. This condition involves analytical tools and the density deviation of the points of X with respect to its average. This last point is controlled by the Lagarias and Pleasants Theorem 6.

Bibliography

[AO95] Jan M. Aarts and Lex G. Oversteegen, *Matchbox manifolds*, Continua (Cincinnati, OH, 1994), Lecture Notes in Pure and Appl. Math., vol. 170, Dekker, New York, 1995, pp. 3–14. MR 1326831 (96d:54034)

[APCC+15] José Aliste-Prieto, Daniel Coronel, María Isabel Cortez, Fabien Durand, and Samuel Petite, *Mathematics of Aperiodic Order*, Progress in Mathematics, ch. Linearly repetitive Delone set, p. 28, Birkhäuser/Springer, Basel, 2015.

[APCG13] José Aliste-Prieto, Daniel Coronel, and Jean-Marc Gambaudo, *Linearly repetitive Delone sets are rectifiable*, Ann. Inst. H. Poincaré Anal. Non Linéaire **30** (2013), no. 2, 275–290. MR 3035977

[Aus88] Joseph Auslander, *Minimal flows and their extensions*, North-Holland Mathematics Studies, vol. 153, North-Holland Publishing Co., Amsterdam, 1988, Notas de Matemática [Mathematical Notes], 122. MR 956049 (89m:54050)

[BBG06] Jean Bellissard, Riccardo Benedetti, and Jean-Marc Gambaudo, *Spaces of tilings, finite telescopic approximations and gap-labeling*, Comm. Math. Phys. **261** (2006), no. 1, 1–41. MR 2193205 (2007c:46063)

[BBL13] Adnene Besbes, Michael Boshernitzan, and Daniel Lenz, *Delone sets with finite local complexity: linear repetitivity versus positivity of weights*, Discrete Comput. Geom. **49** (2013), no. 2, 335–347. MR 3017915

[BK98] D. Burago and B. Kleiner, *Separated nets in Euclidean space and Jacobians of bi-Lipschitz maps*, Geom. Funct. Anal. **8** (1998), no. 2, 273–282. MR 1616135 (99d:26018)

[BK02] ———, *Rectifying separated nets*, Geom. Funct. Anal. **12** (2002), no. 1, 80–92. MR 1904558 (2003h:26020)

[CD08] María Isabel Cortez and Fabien Durand, *Self-similar tiling systems, topological factors and stretching factors*, Discrete Comput. Geom. **40** (2008), no. 4, 622–640. MR 2453331 (2009i:52013)

[CDP10] María Isabel Cortez, Fabien Durand, and Samuel Petite, *Linearly repetitive Delone systems have a finite number of nonperiodic Delone system factors*, Proc. Amer. Math. Soc. **138** (2010), no. 3, 1033–1046. MR 2566569 (2011b:37026)

[CGSY99] Dominique Cerveau, Étienne Ghys, Nessim Sibony, and Jean-Christophe Yoccoz, *Dynamique et géométrie complexes*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 8, Société Mathématique de France, Paris; distributed by American Mathematical Society, Providence, RI, 1999, Papers from the Meeting "State of the Art of the Research of the Société Mathématique de France" held at École Normale Supérieure de Lyon, Lyon, January 1997. MR 1760840 (2001a:37002)

[dB81] N. G. de Bruijn, *Algebraic theory of Penrose's nonperiodic tilings of the plane. I, II*, Nederl. Akad. Wetensch. Indag. Math. **43** (1981), no. 1, 39–52, 53–66. MR 609465 (82e:05055)

[DHS99] F. Durand, B. Host, and C. Skau, *Substitutive dynamical systems, bratteli diagrams and dimension groups*, Ergod. Th. & Dynam. Sys. **19** (1999), 953–993.

[DK78] F. M. Dekking and M. Keane, *Mixing properties of substitutions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **42** (1978), no. 1, 23–33. MR 0466485 (57 #6363)

[dlLW09] R. de la Llave and A. Windsor, *An application of topological multiple recurrence to tiling*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), no. 2, 315–324. MR 2505641 (2010c:37026)

[Dur00] F. Durand, *Linearly recurrent subshifts have a finite number of non-periodic subshift factors*, Ergod. Th. & Dynam. Sys. **20** (2000), 1061–1078.

[Fur81] H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton, N.J., 1981, M. B. Porter Lectures. MR 603625 (82j:28010)

[Gro93] M. Gromov, *Asymptotic invariants of infinite groups*, Geometric group theory, Vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser., vol. 182, Cambridge Univ. Press, Cambridge, 1993, pp. 1–295. MR 1253544 (95m:20041)

[KP00] Johannes Kellendonk and Ian F. Putnam, *Tilings, C^*-algebras, and K-theory*, Directions in mathematical quasicrystals, CRM Monogr. Ser., vol. 13, Amer. Math. Soc., Providence, RI, 2000, pp. 177–206. MR 1798993 (2001m:46153)

[Len02] Daniel Lenz, *Uniform ergodic theorems on subshifts over a finite alphabet*, Ergodic Theory Dynam. Systems **22** (2002), no. 1, 245–255. MR 1889573 (2003j:37008)

[Len04] ———, *Aperiodic linearly repetitive Delone sets are densely repetitive*, Discrete Comput. Geom. **31** (2004), no. 2, 323–326. MR 2060644 (2005a:37027)

[LP02] Jeffrey C. Lagarias and Peter A. B. Pleasants, *Local complexity of Delone sets and crystallinity*, Canad. Math. Bull. **45** (2002), no. 4, 634–652, Dedicated to Robert V. Moody. MR 1941231 (2003i:52023)

[LP03] ———, *Repetitive Delone sets and quasicrystals*, Ergodic Theory Dynam. Systems **23** (2003), no. 3, 831–867. MR 1992666 (2005a:52018)

[McM98] C. T. McMullen, *Lipschitz maps and nets in Euclidean space*, Geom. Funct. Anal. **8** (1998), no. 2, 304–314. MR 1616159 (99e:58017)

[MH40] M. Morse and G. A. Hedlund, *Symbolic dynamics II. Sturmian trajectories*, Amer. J. Math. **62** (1940), 1–42.

[Pet99] Karl Petersen, *Factor maps between tiling dynamical systems*, Forum Math. **11** (1999), no. 4, 503–512. MR 1699171 (2000f:37019)

[Rob96] E. Arthur Robinson, Jr., *The dynamical theory of tilings and quasicrystallography*, Ergodic theory of $\mathbf{Z}^d$ actions (Warwick, 1993–1994), London Math. Soc. Lecture Note Ser., vol. 228, Cambridge Univ. Press, Cambridge, 1996, pp. 451–473. MR 1411233 (97i:52023)

[RS01] Charles Radin and Lorenzo Sadun, *Isomorphism of hierarchical structures*, Ergodic Theory Dynam. Systems **21** (2001), no. 4, 1239–1248. MR 1849608 (2002e:37021)

[Rud89] Daniel J. Rudolph, *Markov tilings of $\mathbf{R}^n$ and representations of $\mathbf{R}^n$ actions*, Measure and measurable dynamics (Rochester, NY, 1987), Contemp. Math., vol. 94, Amer. Math. Soc., Providence, RI, 1989, pp. 271–290. MR 1012996 (91b:28016)

[Sol97] Boris Solomyak, *Dynamics of self-similar tilings*, Ergod. Th. & Dynam. Sys. **17** (1997), 695–738.

[Sol98] B. Solomyak, *Nonperiodicity implies unique composition for self-similar translationally finite tilings*, Discrete Comput. Geom. **20** (1998), no. 2, 265–279. MR 1637896 (99f:52028)

[Sol11] Yaar Solomon, *Substitution tilings and separated nets with similarities to the integer lattice*, Israel J. Math. **181** (2011), 445–460. MR 2773052 (2012c:37030)

[Sol14] ———, *A simple condition for bounded displacement*, J. Math. Anal. Appl. **414** (2014), no. 1, 134–148. MR 3165299