Beyond Pisot dynamics

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Dynamics of Cantor sets, Salta CIMPA school

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A substitution on words : the Fibonacci substitution

Definition A substitution σ is a morphism of the free monoid

Positive morphism of the free group, no cancellations

Example

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 1$$

1
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121
12112
12112121

$$\sigma^{\infty}(1) = 1211212112112 \cdots$$

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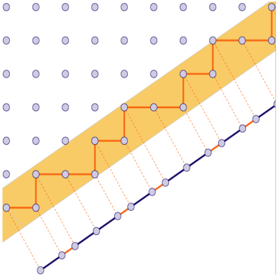


The Fibonacci word is a Sturmian word (cf. V. Delecroix's lecture)

The Fibonacci word yields a quasicrystal

Quasiperiodicity and quasicrystals

Quasicrystals are solids discovered in 84 with an atomic structure that is both ordered and aperiodic [Shechtman-Blech-Gratias-Cahn]

An aperiodic system may have long-range order

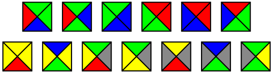
[What is... a Quasicrystal? M. Senechal]

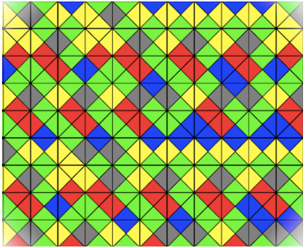
Quasiperiodicity and quasicrystals

Quasicrystals are solids discovered in 84 with an atomic structure that is both ordered and aperiodic [Shechtman-Blech-Gratias-Cahn]

An aperiodic system may have long-range order

(cf. Aperiodic tilings [Wang'61, Berger'66, Robinson'71,...)





Quasiperiodicity and quasicrystals

Quasicrystals are solids discovered in 84 with an atomic structure that is both ordered and aperiodic [Shechtman-Blech-Gratias-Cahn]

An aperiodic system may have long-range order

- Quasicrystals produce a discrete diffraction diagram (=order)
- Diffraction comes from regular spacing and local interactions of the point set Λ (consider the relative positions $\Lambda - \Lambda$)

There are mainly two methods for producing quasicrystals

- Substitutions
- Cut and project schemes

[What is... a Quasicrystal? M. Senechal]

Cut and project schemes

Projection of a “plane” slicing through a higher dimensional lattice

- The order comes from the lattice structure
- The nonperiodicity comes from the irrationality of the normal vector of the “plane”

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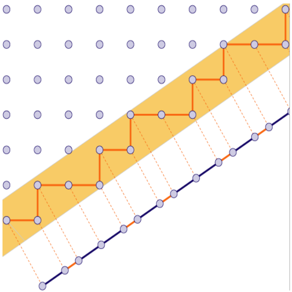
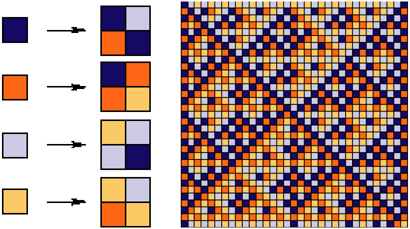


Sturmian words

Substitutions

- Substitutions on words and symbolic dynamical systems
- Substitutions on tiles : inflation/subdivision rules, tilings and point sets



- Tilings Encyclopedia http://tilings.math.uni-bielefeld.de/

[E. Harriss, D. Frettlöh]

A substitution on words : the Fibonacci substitution

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Why the terminology Fibonacci word?

$$\sigma^{n+1}(1) = \sigma^{n}(12) = \sigma^{n}(1)\sigma^{n}(2)$$

$$\sigma^{n}(2) = \sigma^{n-1}(1)$$

$$\sigma^{n+1}(1) = \sigma^{n}(1)\sigma^{n-1}(1)$$

The length of the word $\sigma^{n}(1)$ satisfies the Fibonacci recurrence

Which substitutions do generate quasicrystals?

How to define a notion of order for an infinite word?

Consider the Fibonacci word

$$u = abaababaabaababaababaabaababaabaababaababaa \cdots$$

- There is a simple algorithmic way to construct it

(cf. Kolmogorov complexity)

The complexity of a string is the length of the shortest possible description of the string

But not all substitutions do produce quasicrystals

How to define a notion of order for an infinite word?

Consider the Fibonacci word

$$u = abaababaabaababaababaabaababaabaababaababaa \cdots$$

- There are few local configurations = factors

  A factor is a word made of consecutive occurrences of letters
  ab is a factor, bb is not a factor of the Fibonacci word

  But

  $$\cdots aaaaaaaaaabaaaaaaaaaa \cdots$$

  has as many factors of length n as

  $$\cdots abaababaabaababaababaabaababaabaababaababaa \cdots$$

  The Fibonacci word has $n+1$ factors of length n

How to define a notion of order for an infinite word?

Consider the Fibonacci word

$$u = abaababaabaababaababaabaababaabaababaababaa\cdots$$

- Consider densities of occurrences of factors

  Symbolic discrepancy

  $$\Delta_N = \max_{i \in \mathcal{A}} \left| |u_0 u_1 \dots u_{N-1}|_i - N \cdot f_i \right|$$

  if each letter i has frequency f_i in u

  $$f_i = \lim_{N \to \infty} \frac{|u_0 \cdots u_{N-1}|_i}{N}$$

  The Fibonacci word has bounded symbolic discrepancy

  (cf. good equidistribution properties for real numbers having bounded partial quotients)

1. Prove that every factor W of the Fibonacci word u can be uniquely written as follows:

$$W = A\sigma(V)B,$$

where V is a factor of the Fibonacci word, $A \in \{\varepsilon, a\}$, and $B = a$, if the last letter of W is a, and $B = \varepsilon$, otherwise.

2. Prove that if W is a left special factor distinct from the empty word, then there exists a unique left special factor V such that $W = \sigma(V)B$, where $B = a$, if the last letter of W is a, and $B = \varepsilon$, otherwise. Deduce the general form of the left special factors.

3. Prove that the Fibonacci sequence is not ultimately periodic.

4. Prove that the complexity function of the Fibonacci word is $p_u(n) = n + 1$ for every n.

The Tribonacci substitution [Rauzy'82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

$$\sigma^\infty(1) : \quad 1213121121312121 3 \cdots$$

Its incidence matrix is $M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

The number of i in $\sigma^n(j)$ is given by $M_\sigma^n[i,j]$

Its characteristic polynomial is $X^3 - X^2 - X - 1$

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It is primitive: there exists a power of M_σ which contains only positive entries

$\rightsquigarrow$ Perron-Frobenius theory

one expanding eigendirection

a contracting eigenplane

Pisot number

Pisot-Vijayaraghavan number An algebraic integer is a Pisot number if its algebraic conjugates λ (except itself) satisfy

$$|\lambda| < 1$$

Pisot substitution σ is primitive and its Perron–Frobenius eigenvalue (for its incidence matrix) is a Pisot number

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Pisot + Perron-Frobenius $\rightsquigarrow$ one expanding eigendirection
a contracting eigenplane

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Theorem [Pisot] If $\lambda > 1$ is an algebraic integer, then the distance from λ^n to the nearest integer goes to zero iff λ is a Pisot number

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Fact Words generated by Pisot substitutions have bounded symbolic discrepancy

$$\Delta_N = \max_{i \in \mathcal{A}} \left| |u_0 u_1 \ldots u_{N-1}|_i - N \cdot f_i \right|$$

$$\text{with } f_i = \lim_{N \to \infty} \frac{|u_0 \cdots u_{N-1}|_i}{N}$$

The Pisot substitution conjecture

Substitutive structure + Algebraic assumption (Pisot)

= Order

Symbolic discrepancy

Discrepancy of a sequence

Let $(u_n)_n$ be a sequence with values in $[0,1]$

$$\Delta_N = \limsup_{I \text{ interval}} \left|\{\text{Card}\{0 \le n \le N; u_n \in I\} - N\mu(I)\right|$$

Symbolic discrepancy

Take a sequence $(u_n)_n$ with values in a finite alphabet $\mathcal{A}$

The frequency f_i of a letter i in $u = (u_n)_{n\in\mathbb{N}}$ is defined as the following limit, if it exists

$$f_i = \lim_{n\to\infty} \frac{|u_0 \cdots u_{N-1}|_i}{N}$$

where $|w|_j$ stands for the number of occurrences of the letter j in the factor w

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Assume that each letter i has frequency f_i in u

Symbolic discrepancy

$$\Delta_N = \max_{i\in\mathcal{A}} ||u_0 u_1 \ldots u_{N-1}|_i - N \cdot f_i|$$

Symbolic dynamical system

Let $u = (u_n)$ be an infinite word with values in the finite set $\mathcal{A}$

The symbolic dynamical system generated by u is (X_u, S)

$$X_u := \overline{\{S^n(u);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

This is the set of infinite words whose factors belong to the set of factors of u

Symbolic discrepancies

$$X_u := \overline{\{S^n(u);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

X_u is minimal if $\emptyset$ and X_u are the only closed shift-invariant subsets of X_u

$\rightsquigarrow$ Every infinite word $v \in X_u$ has the same language as u

Symbolic discrepancies

$$X_u := \overline{\{S^n(u);\ \ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

$$\Delta_N = \max_{i \in \mathcal{A}} \left| |u_0 u_1 \ldots u_{N-1}|_i - N \cdot f_i \right|$$

$$\widetilde{\Delta}_N = \limsup_{i \in \mathcal{A},\ k} \left| |u_k \cdots u_{k+N-1}|_i - N \cdot f_i \right|$$

If X_u is minimal

$$\begin{aligned} \widetilde{\Delta}_N = & \quad \limsup\nolimits_{i \in \mathcal{A},\ k} \left| |u_k \cdots u_{k+N-1}|_i - N \cdot f_i \right| \\ = & \quad \limsup\nolimits_{i \in \mathcal{A},\ w \in \mathcal{L}_N(u)} \left| |w|_i - N \cdot f_i \right| \\ = & \quad \limsup\nolimits_{i \in \mathcal{A},\ v \in X_u} \left| |v_0 v_1 \ldots v_{N-1}|_i - N \cdot f_i \right| \end{aligned}$$

$\mathcal{L}_N(u)$ is the set of factors of u of length N

Symbolic discrepancies

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$\mathcal{L}_N(u)$ is the set of factors of u of length N

We can also consider factors w and not only letters

Balancedness

An infinite word $u \in \mathcal{A}^{\mathbb{N}}$ is said to be (finitely) balanced if there exists a constant $C > 0$ such that for any pair of factors of the same length v, w of u, and for any letter $i \in \mathcal{A}$,

$$||v|_i - |w|_i| \leq C$$

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Fibonacci word $\sigma \colon a \mapsto ab,\ b \mapsto a$ $\quad$ σ is called a substitution

$$a$$
$$ab$$
$$aba$$
$$abaab$$
$$abaababa$$

$$\sigma^{\infty}(a) = abaababaabaababaababaabaababaabaababaabab\ldots$$

The factors of length 5 contain 3 or 4 a's

Remark [B. Adamczewski] There exists an infinite word $u \in \{0,1\}^{\mathbb{N}}$ such that

- u has has a frequency vector
- $\Delta_N = O(g(N))$ with $g(N) = o(N)$
- for every integer N, $\widetilde{\Delta}_N = O(N)$

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Take

$$u = 01\,0^{[g(1)]}1^{[g(1)]}\,0101\,0^{[g(2)]}1^{[g(2)]}\,\cdots(01)^n\,0^{[g(n)]}1^{[g(n)]}$$

$$||u_0\cdots u_{N-1}|_i - N/2| \leq 1/2g(N)$$

u is not uniformly recurrent

Equidistribution vs. well-equidistribution

Let u be an infinite word with values in the finite alphabet $\mathcal{A}$

$$\widetilde{\Delta}_N = \limsup_{i \in \mathcal{A},\ k} \left| |u_k \cdots u_{k+N-1}|_i - N \cdot f_i \right|$$

u is well-distributed with respect to letters if $\widetilde{\Delta}_N = o(N)$
$\rightsquigarrow$ uniformly in k

The frequency of a factor w in u is defined as the limit when n tends towards infinity, if it exists, of the number of occurrences of w in $u_0 u_1 \cdots u_{n-1}$ divided by n

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The infinite word u has uniform letter frequencies if, for every factor w of u, the number of occurrences of w in $u_k \cdots u_{k+n-1}$ divided by n has a limit when n tends to infinity, uniformly in k

Balance and equidistribution

An infinite word $u \in \mathcal{A}^{\mathbb{N}}$ is (finitely) balanced if and only if

- it has uniform letter frequencies
- there exists a constant B such that for any factor w of u, we have $||w|_i - f_i|w|| \leq B$ for all letter i in $\mathcal{A}$

where f_i is the frequency of i

Balance and equidistribution

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- there exists a constant B such that for any factor w of u, we have $||w|_i - f_i|w|| \leq B$ for all letter i in $\mathcal{A}$

where f_i is the frequency of i

Proof

Let u be an infinite word with vector frequency f and such that $||w|_i - f_i|w|| \leq B$ for every factor w and all letters i in $\mathcal{A}$.

For every pair of factors w_1 and w_2 with the same length n, we have

$$||w_1|_i - |w_2|_i| \leq ||w_1|_i - nf_i| + ||w_2|_i - nf_i| \leq 2B$$

Hence u is $2B$-balanced

Finite balancedness implies the existence of uniform letter frequencies

Proof Assume that u is C-balanced and fix a letter i

Let N_p be such that for every word of length p of u, the number of occurrences of the letter i belongs to the set

$$\{N_p, N+1, \cdots, N_p + C\}$$

The sequence $(N_p/p)_{p\in\mathbb{N}}$ is a Cauchy sequence. Indeed consider a factor w of length pq

$$pN_q \leq |w|_i \leq pN_q + pC, \qquad qN_p \leq |w|_i \leq qN_p + qC.$$

$$-C/p \leq N_p/p - N_q/q \leq C/q$$

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Let $f_i = \lim N_q/q$

$$-C \leq N_p - pf_i \leq 0 \quad (q \to \infty)$$

Then, for any factor w

$$\left|\frac{|w|_i}{|w|} - f_i\right| \leq \frac{C}{|w|} \qquad \rightsquigarrow \text{uniform frequencies}$$

- Finite balancedness implies the existence of uniform letter frequencies
- If u has letter frequencies, then u is finitely balanced if and only if its discrepancy $\Delta(u)$ is finite

- Let σ be a primitive substitution and λ be its PF eigenvalue.
- Let d' stand for the number of distinct eigenvalues of M_σ.
- Let λ_i, for $i = 1, \cdots, d'$, stand for the eigenvalues of σ, with $\lambda_1 = \lambda$, and let $\alpha_i + 1$ stand for their multiplicities in the minimal polynomial of the incidence matrix M_σ.
- We order them as follows. Let i, k such that $2 \leq i < k \leq d'$.
  If $|\lambda_i| \neq |\lambda_k|$, then $|\lambda_i| > |\lambda_k|$.
  If $|\lambda_i| = |\lambda_k|$, then $\alpha_i \geq \alpha_k$. We also add that if $|\lambda_i| = |\lambda_k| = 1$, and $\alpha_i = \alpha_k$, if λ_i is not a root of unity and λ_k is a root of unity, then $i < k$.

Theorem Primitive Pisot substitutions are balanced, and have finite discrepancy.
Proof Let σ be a primitive Pisot substitution over the alphabet $\mathcal{A}$. Let us prove that σ has finite discrepancy. Let $(f_i)_i$ stand for its letter frequency vector. We consider the abelianization map I defined as the map

$$I \colon \mathcal{A}^* \to \mathbb{N}^d, \; w \mapsto (|w|_1, |w|_2, \cdots, |w|_d).$$

$$I(\sigma(w)) = M_\sigma I(w)$$

We first consider a word w of the form $w = \sigma^n(j)$, for j letter in $\mathcal{A}$. The sequence $(|\sigma^n(j)|_i)_n$ satisfies a linear recurrence provided by the minimal polynomial of M_σ.

$$|\sigma^n(j)|_i = C_{i,j}\lambda^n + O(n^{\alpha_2}|\lambda_2|^n).$$

By applying the Perron–Frobenius Theorem, one checks that there exists C_j such that $C_{i,j} = C_j f_i$ for all i, hence

$$|\sigma^n(j)|_i = C_j f_i \lambda^n + O(n^{\alpha_2}|\lambda_2|^n).$$

$$|\sigma^n(j)|_i = C_j f_i \lambda^n + O(n^{\alpha_2}|\lambda_2|^n).$$

We then deduce from $\sum_i f_i = 1$ that

$$|\sigma^n(j)|_i - f_i|\sigma^n(j)| = O(n^{\alpha_2}|\lambda_2|^n).$$

It remains to check that this result also holds for prefixes of the fixed point u. Indeed, it is easy to prove that any prefix w of u can be expanded as:

$$w = \sigma^k(w_k)\sigma^{k-1}(w_{k-1})\dots w_0,$$

where the w_i belong to a finite set of words.This numeration is called Dumont-Thomas numeration.

Theorem[Adamczewski] Let σ be a primitive substitution. Let u be a fixed point of σ.

- If $|\lambda_2| < 1$, then the discrepancy $\Delta(u)$ is finite.
- If $|\lambda_2| > 1$, then $\Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2} n^{(\log_\lambda |\lambda_2])})$.
- If $|\lambda_2| = 1$, and λ_2 is not a root of unity, then

$$\Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2+1}).$$

If λ_2 is a root of unity, then either

$$\Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2+1}), \text{ or } \Delta_n(u) = (O \cap \Omega)((\log n)^{\alpha_2}).$$

- In particular there exist balanced fixed points of substitutions for which $|\theta_2| = 1$. All eigenvalues of modulus one of the incidence matrix have to be roots of unity.
- Observe that the Thue-Morse word is 2-balanced, but if one considers generalized balances with respect to factors of length 2 instead of letters, then it is not balanced anymore.

Frequencies and measures

$$X_u := \overline{\{S^n(u);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

- Having frequencies is a property of the infinite word u while having uniform frequencies is a property of the associated language or shift X_u

Frequencies and measures

$$X_u := \overline{\{S^n(u);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

- A probability measure μ on X_u is said invariant if $\mu(S^{-1}A) = \mu(A)$ for all measurable subset $A \subset X$
- An invariant probability measure on a shift X is said ergodic if any shift-invariant measurable set has either measure 0 or 1
- The property of uniform frequency of factors for a shift X is equivalent to unique ergodicity: there exists a unique shift-invariant probability measure on X

Frequencies and measures

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- Having frequencies is a property of the infinite word u while having uniform frequencies is a property of the associated language or shift X_u
- Balancedness is a property of the associated shift and may be thought as a strong form of unique ergodicity

Birkhoff sums

Let μ is an ergodic measure on X_u. The Birkhoff Ergodic theorem says that for μ-a.e. x and for $f \in L_1(X_u, \mathbb{R})$

$$\lim_n \frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) = \int f d\mu$$

The mean behaviour along an orbit=
the mean value of f with respect to μ

μ-almost every infinite word in X_u has frequency $\mu[w]$

$$[w] = \{u \in X; u_0 \dots u_{n-1} = w\}$$

but this frequency is not necessarily uniform

If X_u is uniquely ergodic, the unique invariant measure on X_u is ergodic and the convergence is uniform for all words in X_u

Theorem Let u be a recurrent sequence s.t.

$$p_u(n) \leq Cn \quad \forall n$$

Then there exists a finite set F such that, if

$$D = \bigcup_{n \in \mathbb{Z}} S^n F$$

S is one-to-one from $X_u \setminus D$ to $X_u \setminus D$.

Proof One has $p_u(n+1) - p_u(n) \leq C$ for all n.
Since u is recurrent, every word w of length n has at least one left extension
There can be no more than C words of length n which have two or more left extensions.
Let F be the set of infinite words v in X_u such that $S^{-1}v$ has at least two elements.
If the word $w = (w_n)_{n \in \mathbb{N}} \in F$, then there exists $a \neq b$ such that the sequences $aw_0w_1 \ldots$ and $bw_0w_1 \ldots$ belongs to X_u, and hence the word $w_0 \ldots w_n$ has at least two left extensions for every n.
So F has at most C elements.

Spectrum

Eigenvalue Let (X, T) be a topological dynamical system

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Eigenvalue Let (X, T) be a topological dynamical system

T is a homeomorphism acting on the compact space X

$$\mathbb{T} = \mathbb{R}/\mathbb{Z} \qquad R_\alpha \colon \mathbb{T} \mapsto \mathbb{T},\ x \mapsto x + \alpha$$

Spectrum

Eigenvalue Let (X, T) be a topological dynamical system
A non-zero continuous function $f \in \mathcal{C}(X)$ with complex values is an eigenfunction for T if there exists $\lambda \in \mathbb{C}$ such that

$$\forall x \in X, \ f(Tx) = \lambda f(x)$$

Discrete spectrum (X, T) is said to have pure discrete spectrum if its eigenfunctions span $\mathcal{C}(X)$

Spectrum

Eigenvalue Let (X, T) be a topological dynamical system

Example

$$R_\alpha \colon \mathbb{T}/\mathbb{Z} \to \mathbb{T}/\mathbb{Z},\ x \mapsto x + \alpha$$

$$f_k \colon x \mapsto e^{2i\pi kx},\ f_k \circ R_\alpha = e^{2i\pi k\alpha} f_k$$

Spectrum

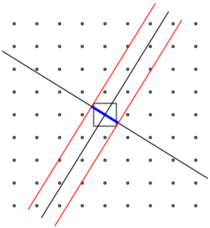
Eigenvalue Let (X, T) be a topological dynamical system

Theorem [Von Neumann] Any invertible and minimal topological dynamical system minimal with topological discrete spectrum is isomorphic to a minimal translation on a compact abelian group

Example In the Fibonacci case $\sigma\colon 1 \mapsto 12, 2 \mapsto 1$

(X_σ, S) is isomorphic to $(\mathbb{R}/\mathbb{Z}, R_{\frac{1+\sqrt{5}}{2}})$

$$\begin{array}{ccc} X_\sigma & \xrightarrow{S} & X_\sigma \\ \downarrow & & \downarrow \\ \mathbb{T} & \xrightarrow[R_\alpha]{} & \mathbb{T} \end{array}$$



The Pisot substitution conjecture

Substitutive structure + Algebraic assumption (Pisot)

= Order (discrete spectrum)

Discrete spectrum = translation on a compact group

Substitutive dynamical systems

Let σ be a primitive substitution over $\mathcal{A}$.
The symbolic dynamical system generated by σ is (X_σ, S)

$$X_\sigma := \overline{\{S^n(u);\ n \in \mathbb{N}\}} \subset \mathcal{A}^{\mathbb{N}}$$

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Question Under which conditions is it possible to give a geometric representation of a substitutive dynamical system as a translation on a compact abelian group? (discrete spectrum)

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The Pisot substitution conjecture Dates back to the 80's

[Bombieri-Taylor, Rauzy,Thurston]

If σ is a Pisot irreducible substitution, then (X_σ, S) has discrete spectrum

Substitutive dynamical systems

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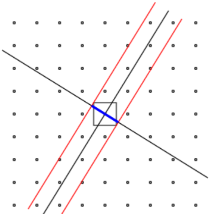
$$R_{\frac{1+\sqrt{5}}{2}} : x \mapsto x + \frac{1+\sqrt{5}}{2} \text{ mod } 1$$



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The Pisot substitution conjecture

If σ is a Pisot irreducible substitution, then (X_σ, S) has discrete spectrum

The conjecture is proved for two-letter alphabets

[Host, Barge-Diamond, Hollander-Solomyak]

Tribonacci's substitution [Rauzy '82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1 \qquad \begin{array}{ccc} X_\sigma & \xrightarrow{S} & X_\sigma \\ \downarrow & & \downarrow \\ \mathbb{T}^2 & \xrightarrow[+(1/\beta,1/\beta^2)]{} & \mathbb{T}^2 \end{array}$$

Question Is it possible to give a geometric representation of the associated substitutive dynamical system X_σ as a Kronecker map $=$ translation on an abelian compact group?

Yes! (X_σ, S) is isomorphic to a translation on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$

Question How to produce explicitly a fundamental domain?

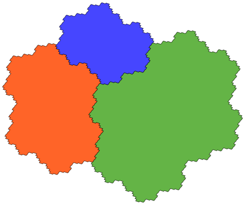
Rauzy fractal G. Rauzy introduced in the 80's a compact set with fractal boundary that tiles the plane which provides a geometric representation of (X_σ, S) $\rightsquigarrow$Thurston for beta-numeration

Tribonacci dynamics and Tribonacci Kronecker map

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

Theorem [Rauzy'82] The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$



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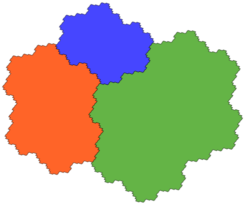

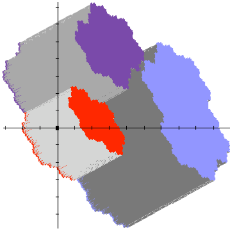

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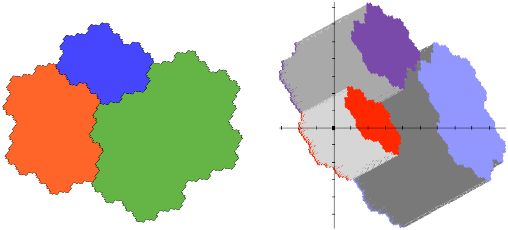


Markov partition for the toral automorphism $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

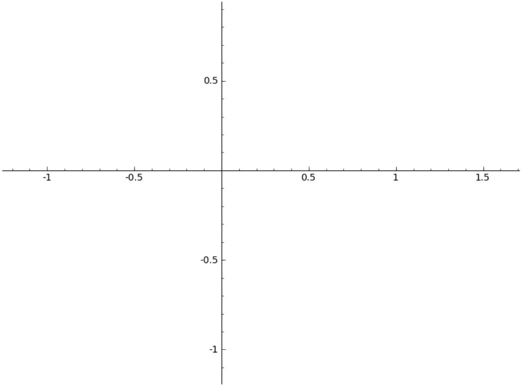
The Rauzy fractal as a geometric representation

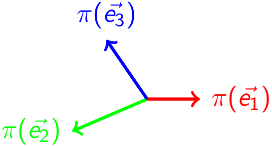
Consider the Tribonacci substitution

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 1$$

$121312112131212131211213\cdots$

π projection along the expanding eigenline onto the contracting plane of the incidence matrix of M_σ





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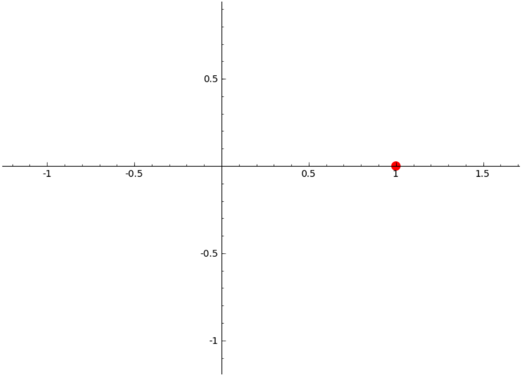
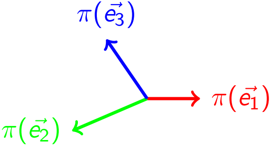
$\pi(\vec{e_1})$



π projection along the expanding eigenline onto the contracting plane of the incidence matrix of M_σ



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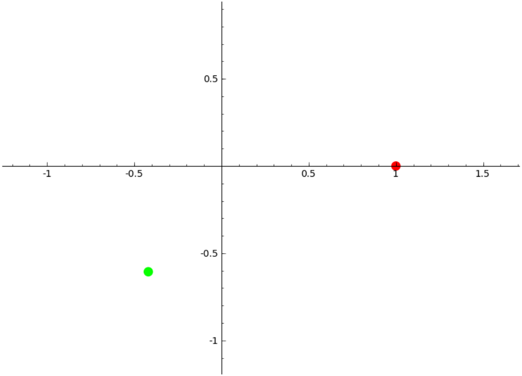

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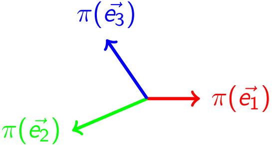

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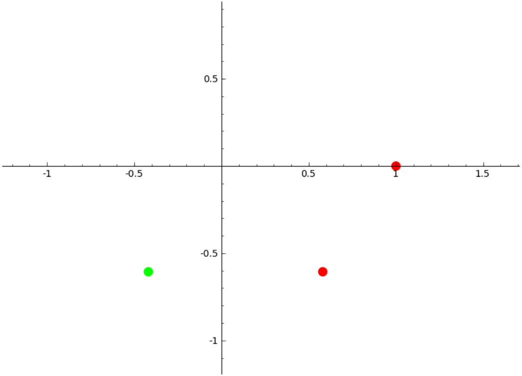

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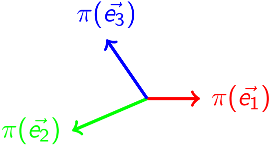

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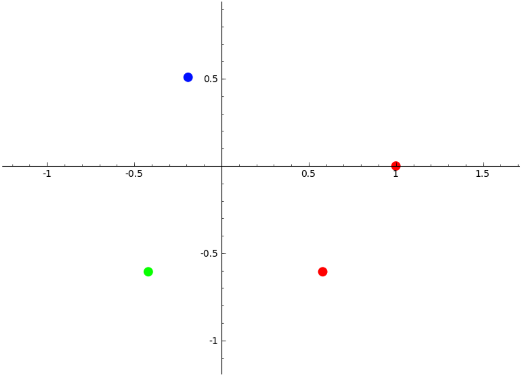

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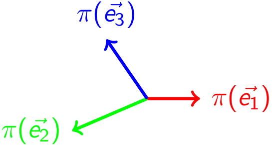

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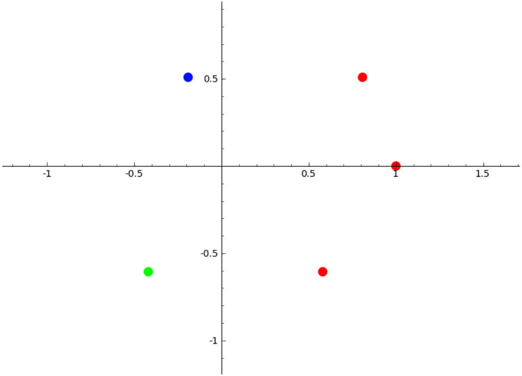

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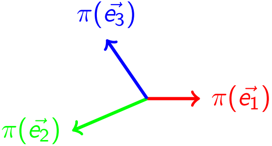

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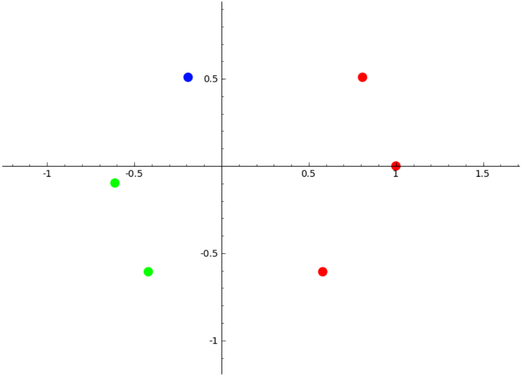

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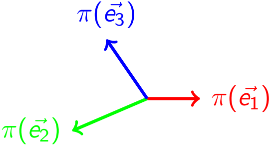

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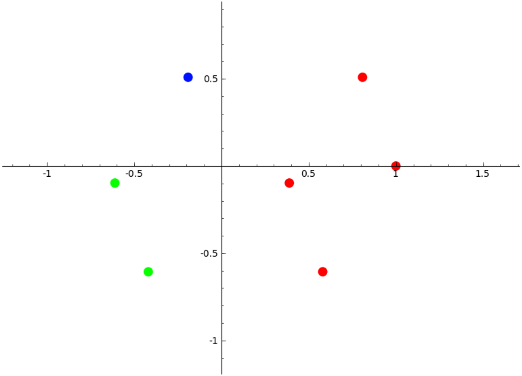

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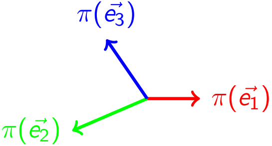

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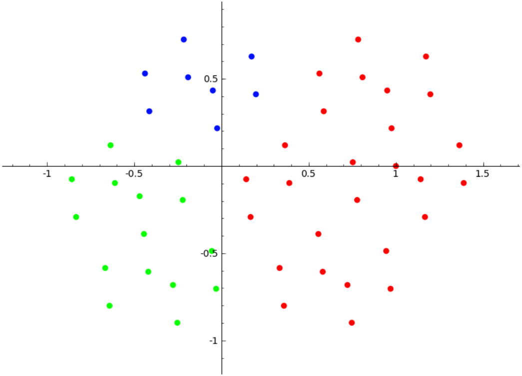

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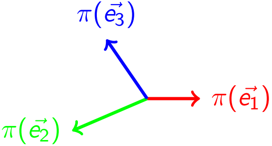

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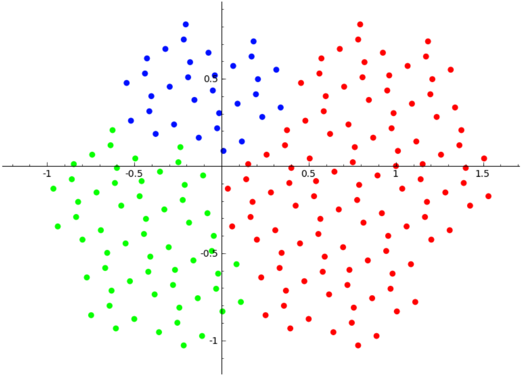

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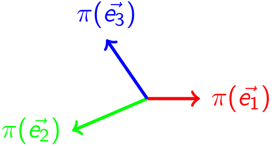

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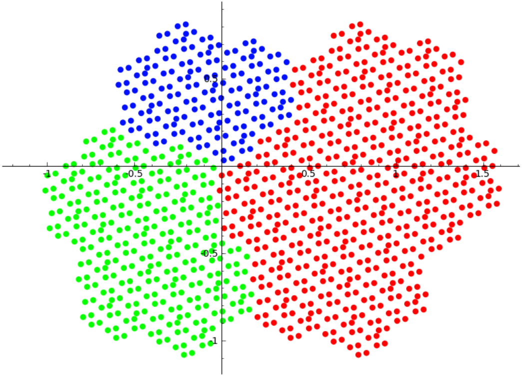

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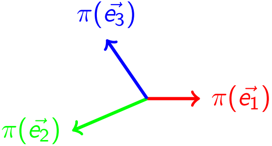

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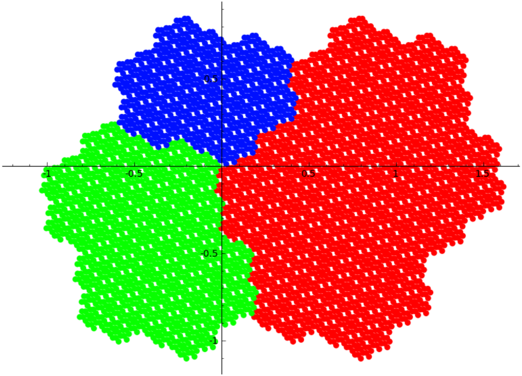

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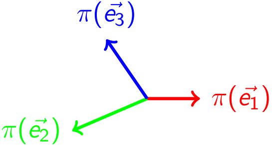

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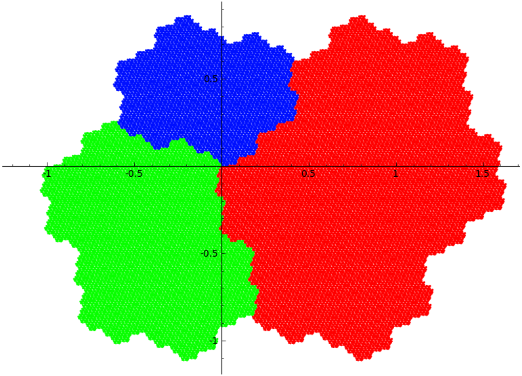

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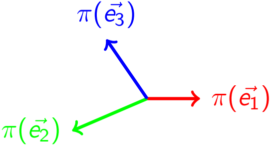
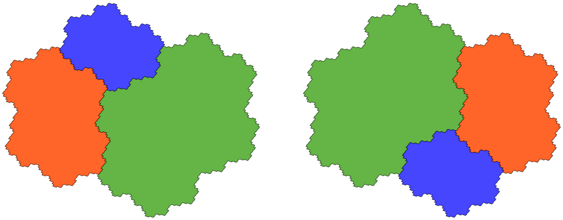


Thanks to T. Jolivet for the slides

Rauzy fractal and dynamics

One first defines an exchange of pieces acting on the Rauzy fractal
The subtiles are disjoint in measure (the proof uses the associated graph-directed Iterated Function System)



π projection along the expanding eigenline onto the contracting plane of the incidence matrix M_σ

The translation vectors are the projections of the canonical basis vectors $\pi(\vec{e_i})$

Rauzy fractal and dynamics

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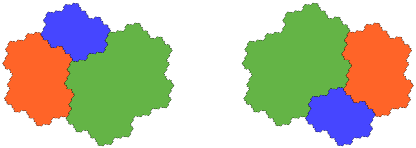
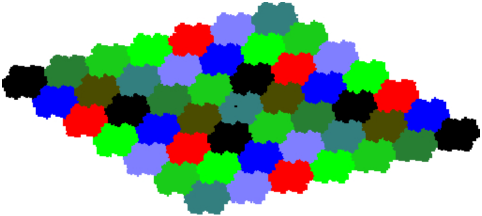
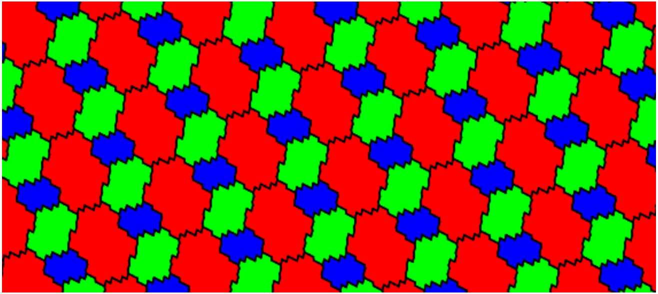


This exchange of pieces factorizes into a translation of $\mathbb{T}^2$
This due to the fact that the Rauzy fractal tiles periodically the plane



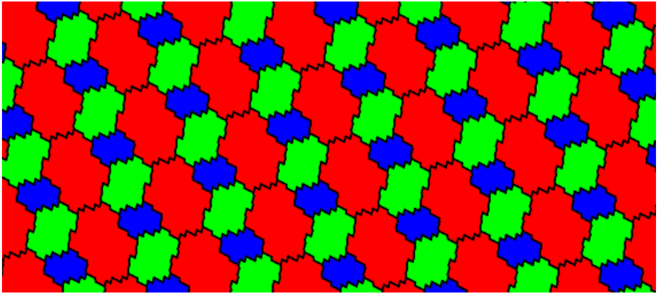
Dynamics of Pisot substitutions

Periodic tiling



Dynamics of Pisot substitutions

Periodic tiling $\longleftrightarrow$ partition of the torus $\mathbb{T}^2$



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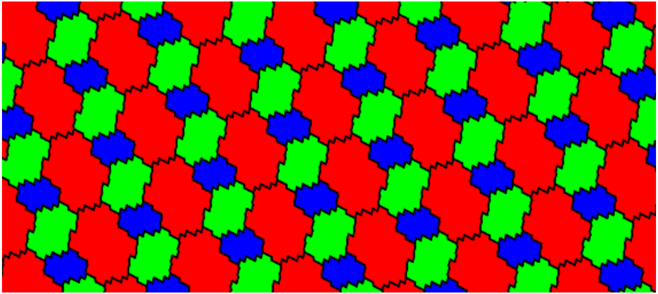

$$(X_\sigma, \text{shift}) \cong (\mathbb{T}^2, x \mapsto x + (\frac{1}{\beta}, \frac{1}{\beta^2}))$$

$$\ldots 1213121121 \ldots \in X_\sigma$$

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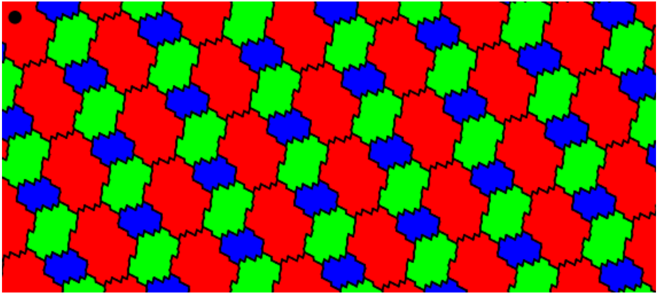

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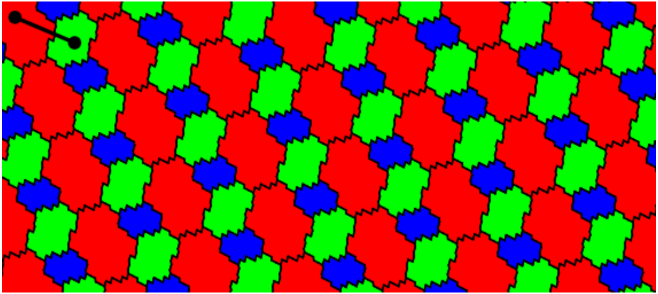

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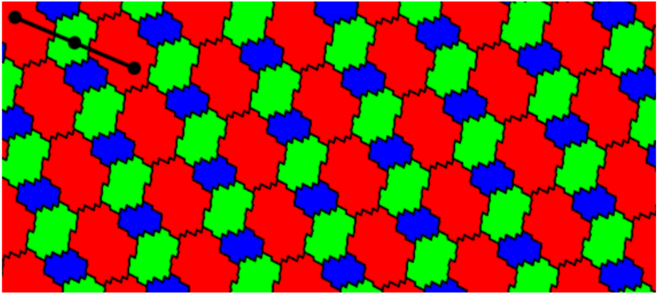
Dynamics of Pisot substitutions

Periodic tiling $\longleftrightarrow$ partition of the torus $\mathbb{T}^2$



$$(X_\sigma, \text{shift}) \cong (\mathbb{T}^2, x \mapsto x + (\frac{1}{\beta}, \frac{1}{\beta^2}))$$

$$\ldots 12\boxed{1}3121121 \ldots \in X_\sigma$$

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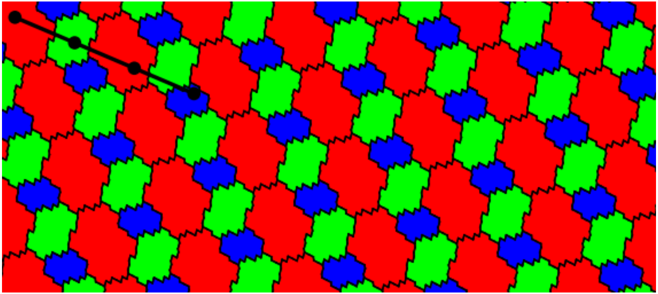

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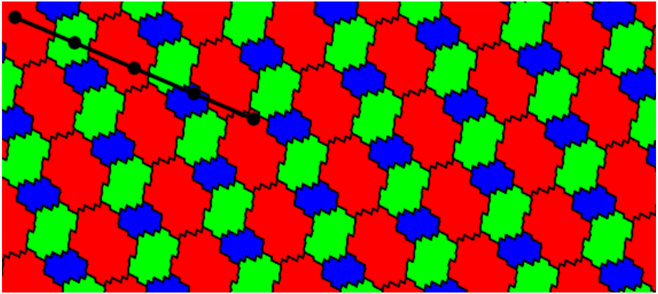

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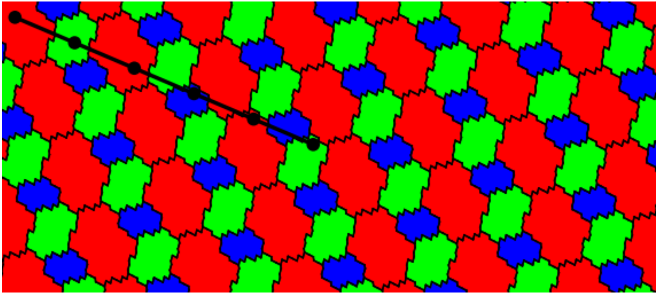

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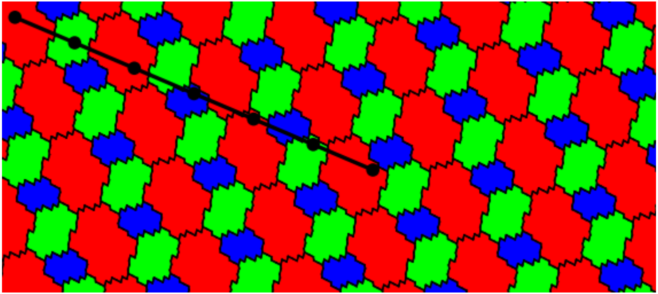

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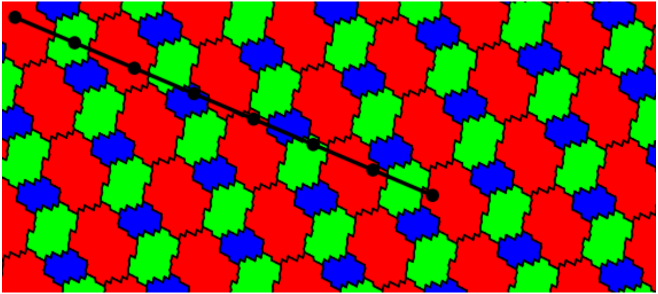

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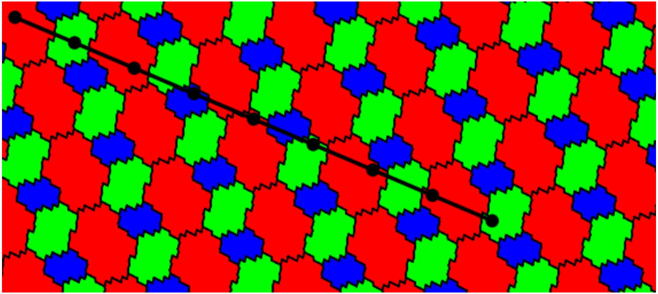

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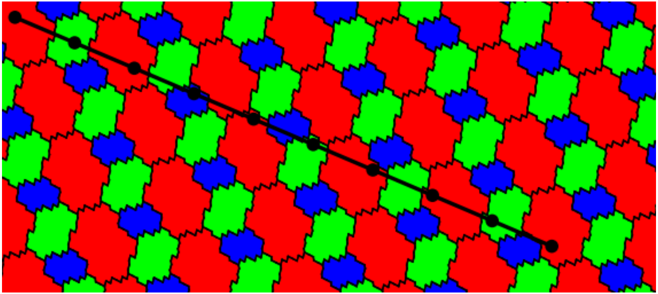

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Why do we get fractals for $d \geq 3$?

- The pieces of the Rauzy fractal are bounded remainder sets
- They produce atoms of Markov partitions for toral automorphisms
- They capture simultaneous approximation properties

Bounded remainder sets and Kronecker sequences

Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in [0, 1]^d$

with $1, \alpha_1, \cdots, \alpha_d$ $\mathbb{Q}$-linearly independent

We consider the Kronecker sequence

$$(\{n\alpha_1\}, \ldots, \{n\alpha_d\})_n$$

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$$R_\alpha \colon \mathbb{T}^d \mapsto \mathbb{T}^d,\ x \mapsto x + \alpha$$

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Bounded remainder set A set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \le n \le N; R_\alpha^n(0) \in X\} - N\mu(X)| \le C$$

Bounded remainder sets

Case $d = 1$

Theorem [Kesten'66] Intervals that are bounded remainder sets are the intervals with length in $\mathbb{Z} + \alpha\mathbb{Z}$

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General dimension d

Theorem [Liardet'87] There are no nontrivial boxes that are bounded remainder sets

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Boxes are not bounded remainder sets

It is possible to find polytopes that are bounded remainder sets for any irrational rotation in any dimension

[Haynes-Koivusalo,Grepstad-Lev]

- Renormalization?
- How well can one approximate a box by bounded remainder sets?

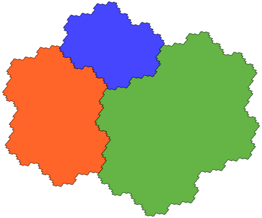
Pisot dynamcis

Bounded remainder set A set X for which there exists $C > 0$ s.t. for all N

$$|\mathrm{Card}\{0 \leq n \leq N; R_\alpha^n(0) \in X\} - N\mu(X)| \leq C$$

$$\sigma\colon 1 \mapsto 12,\ 2 \mapsto 3,\ 3 \mapsto 1$$

Fact The pieces of the Rauzy fractal are bounded remainder sets



Variations around Rauzy fractals

One can define Rauzy fractals for substitutions over

- Delone sets/cut-and-project schemes [Lee,Moody,Solomyak,Sing,Frettlöh,Baake etc.]
- trees [Bressaud-Jullian]
- on the free group [Arnoux-B.-Hillion-Siegel, Coulbois-Hillion]

and for numeration dynamical systems defined in terms of Pisot numbers

- beta-numeration [Thurston, Akiyama, Ei-Ito-Rao,B.-Siegel, Minervino-Steiner, etc.]
- abstract numerations [B.-Rigo]
- Shift Radix Systems [B.-Siegel-Steiner-Surer-Thuswaldner]

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and even

- Selmer numbers [Kenyon-Vershik]
- in codimension 2 [Arnoux-Furukado-Harris-Ito]
- Pisot families [Akiyama-Lee, Barge-Stimac-Williams]
- nonalgebraic parameters $\rightsquigarrow$ S-adic Rauzy fractals

Beyond the Pisot substitution conjecture

How to reach nonalgebraic parameters?

Theorem [Rauzy'82]

$$\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$$

(X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$

$$R_\beta : \mathbb{T}^2 \to \mathbb{T}^2,\ x \mapsto x + (1/\beta, 1/\beta^2)$$

- We want to find symbolic realizations for toral translations
- We want to reach nonalgebraic parameters
- We consider not only one substitution

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- We want to find symbolic realizations for toral translations
- We want to reach nonalgebraic parameters by considering convergent products of matrices
- We consider not only one substitution but a sequence of substitutions Non-stationary dynamics

$\rightsquigarrow$ Multidimensional continued fractions algorithms/Generalized Euclid algorithms

S-adic words

S-adic expansions

- Let $\mathcal{S}$ be a set S of substitutions
- Let $s = (\sigma_n)_{n\in\mathbb{N}} \in S^{\mathbb{N}}$, with $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$, be a sequence of substitutions
- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of letters with $a_n \in \mathcal{A}_n$ for all n

We say that the infinite word $u \in \mathcal{A}^{\mathbb{N}}$ admits $((\sigma_n, a_n))_n$ as an S-adic representation if

$$u = \lim_{n\to\infty} \sigma_0\sigma_1\cdots\sigma_{n-1}(a_n)$$

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The terminology comes from Vershik adic transformations
Bratteli diagrams

S stands for substitution, adic for the inverse limit
powers of the same substitution= partial quotients

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- The sequence s is called the directive sequence and the sequences of letters $(a_n)_n$ will only play a minor role compared to the directive sequence.
- If the set S is finite, it makes no difference to consider a constant alphabet (i.e., $\mathcal{A}_n^* = \mathcal{A}^*$ for all n and for all substitution σ in S).

First remarks

- Without further restrictions, to be S-adic is not a property of the sequence but a way to construct it
- An S-adic representation defined by the directive sequence $(\sigma_n)_{n\in\mathbb{N}}$ is everywhere growing if for any sequence of letters $(a_n)_n$, one has

$$\lim_{n\to+\infty} |\sigma_0\sigma_1\cdots\sigma_{n-1}(a_n)| = +\infty$$

- Substitutions are non-erasing: the image of any letter is different from the empty word

Every sequence is S-adic [Cassaigne]

Let $u = u_0 u_1 u_2 \cdots \in \mathcal{A}^{\mathbb{N}}$. Consider the alphabet $\mathcal{A} \cup \{\ell\}$. Let

$$\sigma_a(b) = b, \forall b \in \mathcal{A},\ \sigma_a(\ell) = \ell a$$

$$\tau_{u_0}(a) = a, \forall a \in \mathcal{A},\ \tau(\ell) = u_0.$$

One has

$$u = \lim_{n \to +\infty} \tau_{u_0} \circ \sigma_{u_1} \circ \sigma_{u_2} \circ \cdots \circ \sigma_{u_n}(\ell)$$

It is not everywhere growing

$$|\tau_{u_0} \circ \sigma_{u_1} \circ \cdots \circ \sigma_{u_n}(\ell)| \to \infty$$

but for all $a \in \mathcal{A}$ and for all n

$$|\tau_{u_0} \circ \sigma_{u_1} \circ \cdots \circ \sigma_{u_n}(a)| = 1$$

Dictionary

- S-adic expansion
- Unique ergodicity
- Linear recurrence
- Balance and Pisot properties

- Continued fraction
- Convergence
- Bounded partial quotients
- Strong convergence

Examples

Sturmian words

$\mathcal{A} = \{a, b\}$

$$\tau_a \colon a \mapsto a, b \mapsto ab, \qquad \tau_b \colon a \mapsto ba, b \mapsto b$$

Let $(i_n) \in \{a, b\}^{\mathbb{N}}$. The following limits

$$u = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1}\cdots\tau_{i_{n-1}}(a) = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1}\cdots\tau_{i_{n-1}}(b)$$

exist and coincide whenever the directive sequence $(i_n)_n$ is not ultimately constant.

This latter condition is equivalent to the everywhere growing property.

The infinite words thus produced belong to the class of Sturmian words.

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More generally, a Sturmian word is an infinite word whose set of factors coincides with the set of factors of a sequence of the previous form, with the sequence $(i_n)_{n\geq 0}$ being not ultimately constant.

Sturmian words and continued fractions

The incidence matrix of σ is the square matrix $M_\sigma = (m_{i,j})_{i,j}$ with entries $m_{i,j} := |\sigma(j)|_i$. It is a non-negative integer matrix.

$$\tau_a\colon a \mapsto a, b \mapsto ab, \qquad \tau_b\colon a \mapsto ba, b \mapsto b$$

$$M_{\tau_a} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad M_{\tau_b} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$u = \lim_{n\to\infty} \tau_{i_0}\tau_{i_1}\cdots\tau_{i_{n-1}}(a)$$

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Sturmian words and continued fractions

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There exists $\alpha \in (0,1)$ such that limit cone satisfies

$$\bigcap_n M_{\tau_{i_0}}\cdots M_{\tau_{i_n}}\mathbb{R}^d_+ = \mathbb{R}^+ \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}$$

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with the directive sequence $(i_n)_n$ being not ultimately constant.

The frequency of a letter i in u is defined as the limit when n tends towards infinity, if it exists, of the number of occurrences of i in $u_0 u_1 \cdots u_{n-1}$ divided by n.

$$\bigcap_n M_{\tau_{i_0}} \cdots M_{\tau_{i_n}} \mathbb{R}_+^d = \mathbb{R}^+ \begin{bmatrix} \alpha \\ 1-\alpha \end{bmatrix}$$

α is the frequency of a's and the sequence (i_n) is produced by the continued faction expansion of α

Arnoux-Rauzy words

- Let $\mathcal{A} = \{1, 2, \ldots, d\}$. We define the Arnoux-Rauzy substitutions as

$$\mu_i : \ i \mapsto i, \ j \mapsto ji \text{ for } j \in \mathcal{A} \setminus \{i\}\,.$$

- An Arnoux-Rauzy word is an infinite word $\omega \in \mathcal{A}^{\mathbb{N}}$ whose set of factors coincides with the set of factors of a sequence of the form

$$\lim_{n\to\infty} \mu_{i_0}\mu_{i_1}\cdots\mu_{i_n}(1),$$

where the sequence $(i_n)_{n\geq 0} \in \mathcal{A}^{\mathbb{N}}$ is such that every letter in $\mathcal{A}$ occurs infinitely often in $(i_n)_{n\geq 0}$.

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- $d = 3$

$$\mu_1 : \begin{array}{ll} 1 & \mapsto 1 \\ 2 & \mapsto 21 \\ 3 & \mapsto 31 \end{array} \qquad \mu_2 : \begin{array}{ll} 1 & \mapsto 12 \\ 2 & \mapsto 2 \\ 3 & \mapsto 32 \end{array} \qquad \mu_3 : \begin{array}{ll} 1 & \mapsto 13 \\ 2 & \mapsto 23 \\ 3 & \mapsto 3 \end{array}$$

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- Equivalent definition
  - $p(n) = (d-1)n + 1$ factors of length n for every n
  - one right and one left special factor of each length (w right special$=w$ has several extensions: wa and wb factors with $a \neq b$)

Arnoux-Rauzy words

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$$u = \lim_{n \to \infty} \mu_{i_0} \mu_{i_1} \cdots \mu_{i_n}(1)$$

and every letter in $\{1, 2, 3\}$ occurs infinitely often in $(i_n)_{n \geq 0}$

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Example The Tribonacci substitution and its fixed point

Arnoux-Rauzy words

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- The set of the letter density vectors of AR words has zero measure

- They code particular systems of isometries (pseudogroups of rotations) [Arnoux-Yoccoz, Novikov, Dynnikov-De Leo, Levitt -Yoccoz, etc.]

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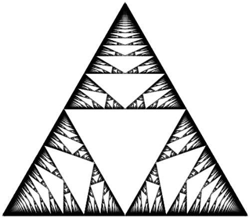

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- There exist AR words that do not have bounded symbolic discrepancy [Cassaigne-Ferenczi-Messaoudi]

- There exist AR words that are (measure-theoretically) weak mixing [Cassaigne-Ferenczi-Messaoudi]

S-adic expansions and factor complexity

Let X be a symbolic dynamical system.
Let $p_X(n)$= number of factors of length n (factor complexity)

Theorem [Cassaigne] A symbolic dynamical system X has at most linear complexity

$$\exists C,\ p_X(n) \leq CN,\ \forall n$$

if and only if $p_X(n+1) - p_X(n)$ is bounded

Theorem [Ferenczi] Let X be a minimal symbolic system on a finite alphabet $\mathcal{A}$ such that its complexity function $p_X(n)$ is at most linear
Then u admits an everywhere growing S-adic representation

See also [Durand, Leroy, Richomme]

Alphabet growth and entropy [T. Monteil]

Theorem Let $(\sigma_n)_n$ be a sequence of substitutions, with $\sigma_n : \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$, and let X be the associated S-adic shift. Let

$$\beta_n^- = \min_{a \in \mathcal{A}_n} |\sigma_0 \circ \sigma_1 \circ \cdots \circ \sigma_{n-1}(a)|.$$

Then, the toplogical entropy h_X of X satisfies

$$h_X \leq \inf_{n \geq 0} \frac{\log \operatorname{Card} \mathcal{A}_n}{\beta_n^-}.$$

In particular, if $(\sigma_n)_n$ is everywhere growing and the alphabets $\mathcal{A}_n$ are of bounded cardinality, then X has zero entropy.

Proof

Let n be fixed. Let $W_n = \{\sigma_0 \dots \sigma_{n-1}(i) \mid i \in \mathcal{A}_n\}$ and let $\beta_n^+ = \max_{i \in \mathcal{A}_n} |\sigma_{[0,n)}(i)|$. By definition, any factor w in X can be decomposed as $w = p v_1 \dots v_k s$ where the v_j belong to W_n, p is a suffix of an element of W_n and s a prefix. For any N large enough, any factor w of length N is a factor of a concatenation of at most $\frac{N}{\beta_n^-} + 2$ words in W_n (we include p and s). By taking into account the possible prefixes, there are at most $(\operatorname{Card} \mathcal{A}_n)^{\frac{N}{\beta_n^-}+2} \cdot (\beta_n^+)$ words of length N, which gives

$$\frac{\log p_X(N)}{N} \leq \inf_{n \geq 0} \left(\left(\frac{1}{\beta_n^-} + \frac{2}{N} \right) \log \operatorname{Card} \mathcal{A}_n + \frac{\log \beta_n^+}{N} \right).$$

S-adic conjecture

- Everywhere growing S-adic representations with bounded alphabets only provide words with zero entropy.
- A restriction on S-adic representations yielding to linear complexity cannot be formulated uniquely in terms of the set S of substitutions: there exist sets of substitutions which produce infinite words that have at most linear complexity function, or not, depending on the directive sequences.

S-adicity and complexity [Durand-Leroy-Richomme]

Let $S = \{\sigma, \tau\}$ with

$$\sigma : a \mapsto aab,\ b \mapsto b, \quad \tau : a \mapsto ab,\ b \mapsto ba$$

- τ is the Thue-Morse substitution
- σ has quadratic complexity

Let $(k_n)_n$ be a sequence of non-negative integers, and let u be the S-adic word

$$u = \lim_{n \to \infty} \sigma^{k_0} \tau \sigma^{k_1} \tau \cdots \tau \sigma^{k_n}(a).$$

Then, the S-adic word u has linear factor complexity if and only if the sequence $(k_n)_n$ is bounded

S-adic conjecture

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S-adic conjecture

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- A restriction on S-adic representations yielding to linear complexity cannot be formulated uniquely in terms of the set S of substitutions: there exist sets of substitutions which produce infinite words that have at most linear complexity function, or not, depending on the directive sequences
- The S-adic conjecture thus consists in providing a characterization of the class of S-adic expansions that generate only words with linear factor complexity by formulating a suitable set of conditions on the set S of substitutions together with the associated directive sequences.

S-adicity and complexity [Cassaigne]

There exists an S-adic sequence with an S-adic expansion having

- bounded partial quotients (every substitution comes back with bounded gaps in the S-adic expansion),
- with each substitution being primitive

whose complexity is quadratic

S-adicity and complexity [Cassaigne]

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whose complexity is quadratic

Let

$$\sigma : \ a \mapsto aab, \ b \mapsto b, \ \mu : \ a \mapsto b, \ b \mapsto a.$$

$$u = \lim_n \ \sigma \circ \mu \circ \sigma^2 \circ \mu \circ \sigma^3 \circ \mu \circ \sigma^4 \circ \cdots \circ \sigma^n \circ \mu(b)$$

One has

$$u = \lim_{n \to +\infty} (\sigma \circ \mu \circ \sigma) \circ (\sigma \circ \mu \circ \sigma) \circ \sigma \circ (\sigma \circ \mu \circ \sigma) \cdots \circ (\sigma \circ \mu \circ \sigma) \circ \sigma^n \circ (\sigma \circ \mu \circ \sigma) \cdots$$

The substitution σ has quadratic complexity and the substitution $\sigma \circ \mu \circ \sigma$ is primitive

The substitutions $\sigma \circ \mu$ and $\mu \circ \sigma$ are primitive and appear with bounded gaps

The complexity of u is quadratic

Primitivity and recurrence

Primitivity

The incidence matrix of σ is the square matrix $M_\sigma = (m_{i,j})_{i,j}$ with entries $m_{i,j} := |\sigma(j)|_i$. It is a non-negative integer matrix.

- An S-adic expansion is said weakly primitive if for each n, there exists r such that the substitution $\sigma_n \cdots \sigma_{n+r}$ is positive.
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Minimality and weak primitivity

If σ is a primitive substitution, then the dynamical system (X_σ, T) is minimal

Theorem An infinite word u is uniformly recurrent (or the shift X_u is minimal) if and only if it admits a weakly primitive S-adic representation.

Minimality and weak primitivity

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Theorem An infinite word u is uniformly recurrent (or the shift X_u is minimal) if and only if it admits a weakly primitive S-adic representation.

Proof Let us prove that an S-adic word with weakly primitive expansion is minimal. Let $(\sigma_n)_n$ be weakly primitive. It is everywhere growing. Consider a factor w of the language. It occurs in some $\sigma_{[0,n)}(i)$ for some integer $n \geq 0$ and some letter $i \in \mathcal{A}$. By definition of weak primitivity, there exists an integer r such that $\sigma_{[n,n+r)}$ is positive. Hence w appears in all images of letters by $\sigma_{[0,n+r)}$ which implies uniform recurrence.

Return words

Let u be a given recurrent word (every factor occurs with bounded gaps) and let w be a factor of u.

A return word over w is a word v such that vw is a factor of u, w is a prefix of vw and w has exactly two occurrences in vw

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Example Fibonacci word

$a|ba|a|ba|ba|a|ba|a|ba|ba|a|ba|ba|a|ba|a|ba|ba|a|ba|a|ba|ba|a|ba\cdots$

a and ab are return words to a

S-adic expansions by return words

- Let u be a uniformly recurrent word on $\mathcal{A}_0$ (every factor occurs with bounded gaps)
- Let w be a non-empty factor of u.
- A return word of w is a word separating two successive occurrences of the word w in u (possibly with overlap).
- By coding the initial word u with these return words, one obtains an infinite word called the derived word, defined on a finite alphabet, and still uniformly recurrent.

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- By coding the initial word u with these return words, one obtains an infinite word called the derived word, defined on a finite alphabet, and still uniformly recurrent.
- Indeed, start with the letter u_0.
- There exist finitely many return words to u_0. Let w_1, $w_2, \dots, w_{d_1}$ be these return words, and consider the associated morphism $\sigma_0 : \mathcal{A}_1 \to \mathcal{A}_0,\ i \mapsto w_i$, with $\mathcal{A}_1 = \{1, \dots, d_1\}$. Then, there exists a unique word u' on $\mathcal{A}_1$ such that $u = \sigma_0(u')$. Moreover, u' is uniformly recurrent.
- It is hence possible to repeat the construction and one obtains an S-adic representation of u.

S-adic expansions by return words

- The alphabets of this representation are a priori of unbounded size.
- In the particular case where u is a primitive substitutive word, then the set of derived words is finite. This is even a characterization [Durand]

Theorem A uniformly recurrent word is substitutive if and only if the set of its derived words is finite.

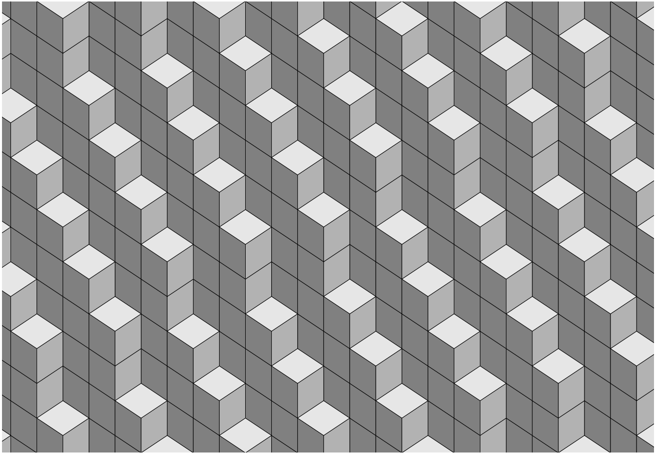
Minimality and weak primitivity

Theorem An infinite word u is uniformly recurrent (or the shift X_u is minimal) if and only if it admits a weakly primitive S-adic representation.

Proof

Conversely, let u be a uniformly recurrent sequence on $\mathcal{A} = \mathcal{A}_0$. Recall that a return word of a factor w is a word separating two successive occurrences of the factor w in u. Let us code the initial word u with these return words; one obtains an infinite word u' on a finite alphabet that is still uniformly recurrent (u' is a derived word). By repeating the construction, one obtains an S-adic representation of u. That S-adic expansion is weakly primitive.

Tilings



Repetitivity

Fact Arithmetic discrete planes are repetitive (factors occur with bounded gaps)

Recurrence function Let N be the smallest integer N such that every square factor of radius N contains all square factors of size n. We set $R(n) := N$.

Linear recurrence There exists C such that $R(n) \leq Cn$ for all n.

Discrete planes [A. Haynes, H. Koivusalo, J. Walton] Linearly recurrent discrete planes are the planes that have a badly approximable normal vector

$$|(r,s)|^2 ||r\alpha + s\beta|| \geq C \text{ for all } (r,s) \neq 0,\ (r,s) \in \mathbb{Z}^2$$

Strong primitivity

Theorem [Durand] Let S be a finite set of substitution and u be an S-adic word having a strongly primitive S-adic expansion. Then, the associated shift (X_u, T) is minimal (that is, u is uniformly recurrent), uniquely ergodic, and it has at most linear factor complexity.

Remark If S is a set of substitutions and $\tau \in S$ is positive, the infinite word generated by a directive sequence for which τ occurs with bounded gaps is uniformly recurrent and has at most linear factor complexity.

LR and S-adicity

Theorem [F. Durand]

- LR implies strongly primitive S-adic
- A strongly primitive S-adic subshift is not necessarily an LR subshift

LR and S-adicity

Theorem [F. Durand]

- LR implies strongly primitive S-adic
- A strongly primitive S-adic subshift is not necessarily an LR subshift

Proof

$$\sigma\colon a \mapsto acb,\ b \mapsto bab,\ c \mapsto cbc$$

$$\tau\colon a \mapsto abc,\ b \mapsto acb,\ c \mapsto aac$$

We consider the S-adic expansion

$$v := \lim_{n\to+\infty} \sigma \circ \tau \circ \sigma^2 \circ \tau \circ \cdots \circ \sigma^n \tau(a)$$

The sequence v is primitive S-adic, it is not LR, it has linear complexity

[F. Durand, "LR Subshsifts have a finite number of non-periodic factors"]

- LR is equivalent with primitive and proper S-adic

Frequencies and invariant measures

Invariant measures

We are given a directive sequence $(\sigma_n)_n$

$$M_{[0,n)} = M_0 M_1 \dots M_{n-1}$$

The limit cone determined by the incidence matrices of the substitutions σ_n is defined as

$$\bigcap_n M_{[0,n)} \mathbb{R}_+^d$$

Invariant measures

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It is the convex hull of the set of half lines $\mathbb{R}_+ f$ generated by the letter frequency vectors f of infinite words in the S-adic shift X

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Theorem [Furstenberg] Let $(M_n)_n$ be a sequence of non-negative integer matrices. Assume that there exists a strictly positive matrix B and indices

$$j_1 < k_1 \leq j_2 < k_2 \leq \cdots$$

such that

$$B = M_{j_1} \cdots M_{k_1 - 1} = M_{j_2} \cdots M_{k_2 - 1} = \cdots$$

Then,

$$\bigcap_n M_{[0,n)} \mathbb{R}_+^d = \mathbb{R}_+ f \quad \text{for some positive vector } f \in \mathbb{R}_+^d.$$

Invariant measures

Theorem Let X be an S-adic shift with directive sequence $\tau = (\tau_n)_n$ where $\tau_n \colon \mathcal{A}_{n+1}^* \to \mathcal{A}_n^*$ and $\mathcal{A}_0 = \{1, \ldots, d\}$. Denote by $(M_n)_n$ the associated sequence of incidence matrices.

If the sequence $(\tau_n)_n$ is everywhere growing, then X has uniform letter frequencies if and only if if the cone $C^{(0)}$ is one-dimensional.

If furthermore, for each k, the limit cone

$$C^{(k)} = \bigcap_{n \to \infty} M_{[k,n)} \mathbb{R}_+^d$$

is one-dimensional, then the S-adic dynamical system (X, T) is uniquely ergodic.

cf. [Bezuglyi, Kwiatkowski, Medynets, Solomyak]
for Bratelli diagrams

Simultaneous approximation and cone convergence

Let f be the generalized eigenvector for an S-adic system on the alphabet $\mathcal{A} = \{1, \ldots, d\}$, normalized by $f_1 + \ldots + f_d = 1$. Let $(e_1, \ldots, e_d)$ be the canonical basis of $\mathbb{R}^d$. Let $(M_n)_n$ stand for the sequence of incidence matrices associated with its directive sequence, and note $A_n = M_0 \cdots M_{n-1}$.

The S-adic system X is weakly convergent toward the non-negative half-line directed by f if

$$\forall i \in \{1, \ldots, d\}, \qquad \lim_{n\to\infty} \mathrm{d}\left(\frac{A_n e_i}{\|A_n e_i\|_1}, f\right) = 0.$$

It is said to be strongly convergent if for a.e. f

$$\forall i \in \{1, \ldots, d\}, \qquad \lim_{n\to\infty} \mathrm{d}(A_n e_i, \mathbb{R}f) = 0.$$

Continued fractions

From S-adic systems to multidimensional continued fractions

Finding an S-adic description of a minimal symbolic dynamical system $\rightsquigarrow$ a multidimensional continued fraction algorithm that governs its letter frequency vector.

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Finding an S-adic description of a minimal symbolic dynamical system $\rightsquigarrow$ a multidimensional continued fraction algorithm that governs its letter frequency vector.

Conversely, we can decide to start with a multidimensional continued fraction algorithm and associate with it an S-adic system. We then translate a continued fraction algorithm into S-adic terms.

Our strategy

- We apply a multidimensional continued fraction algorithm to the line in $\mathbb{R}^3$ directed by a given vector $\mathbf{u} = (u_1, u_2, u_3)$
- We then associate with the matrices produced by the algorithm substitutions, with these substitutions having the matrices produced by the continued fraction algorithm as incidence matrices

Multidimensional continued fractions

If we start with two parameters (α, β), one looks for two rational sequences (p_n/q_n) et (r_n/q_n) with the same denominator that satisfy

$$\lim p_n/q_n = \alpha, \lim r_n/q_n = \beta.$$

Continued fractions

- Euclid's algorithm Starting with two numbers, one subtracts the smallest to the largest
- Unimodularity

$$\det \begin{bmatrix} p_{n+1} & q_{n+1} \\ p_n & q_n \end{bmatrix} = \pm 1$$

Rem $SL(2, \mathbb{N})$ is a finitely generated free monoid. It is generated by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

- Best approximation property

Theorem A rational number p/q is a best approximation of the real number α if every p'/q' wth $1 \leq q' \leq q$, $p/q \neq p'/q'$ satifies

$$|q\alpha - p| < |q'\alpha - p'|$$

Every best approximation of α is a convergent

From $SL(2, \mathbb{N})$ to $SL(3, \mathbb{N})$

- $SL(2, \mathbb{N})$ is a free and finitely generated monoid
- $SL(3, \mathbb{N})$ is not free
- $SL(3, \mathbb{N})$ is not finitely generated. Consider the family of matrices

$$\begin{pmatrix} 1 & 0 & n \\ 1 & n-1 & 0 \\ 1 & 1 & n-1 \end{pmatrix}$$

These matrices are undecomposable for $n \geq 3$ [Rivat]

Multidimensional continued fractions

There is no canonical generalization of continued fractions to higher dimensions

Several approaches are possible

- best simultaneous approximations but we then loose unimodularity, and the sequence of best approximations heavily depends on the chosen norm [Lagarias]
- Klein polyhedra and sails [Arnold]
- unimodular multidimensional Euclid's algorithms
  - Fibered systems e.g., Jacobi-Perron algorithm, Brun algorithm [Brentjes, Schweiger]
  - sequences of nested cones approximating a direction [Nogueira]
  - lattice reduction / geodesic flow (LLL), [Lagarias],[Ferguson-Forcade], [Just], [Grabiner-Lagarias][Smeets]

What is expected?

We are given $(\alpha_1, \cdots, \alpha_d)$ which produces a sequence of basis $(B^{(k)})$ of $\mathbb{Z}^{d+1}$ and/or a sequence of approximations $(p_1^{(k)}), \cdots, p_d^{(k)}, q^{(k)})$

Arithmetics A two-dimensional continued fraction algorithm is expected to

- detect integer relations for $(1, \alpha_1, \cdots, \alpha_d)$
- give algebraic characterizations of periodic expansions
- converge sufficiently fast

$$\max_i \operatorname{dist}(b_i^{(k)}, (\alpha, 1)\mathbb{R}) \to_k 0$$

- and provide good rational approximations

Good means "with respect to Dirichlet's theorem": there exist infinitely many $(p_i/q)_{1 \le i \le d}$ such that

$$\max_i |\alpha_i - p_i/q| \le \frac{1}{q^{1+1/d}}$$

Examples of multidimensional Euclid's algorithms

- Jacobi-Perron: we subtract the first one to the two other ones with $0 \leq x_1, x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_2 - [\frac{x_2}{x_1}]x_1, x_3 - [\frac{x_3}{x_1}]x_1, x_1)$$

- Brun: we subtract the second largest and we reorder with $x_1 \leq x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - x_2)$$

- Poincaré: we subtract the previous one and we reorder with $x_1 \leq x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2 - x_1, x_3 - x_2)$$

- Selmer: we subtract the smallest to the largest and we reorder with $x_1 \leq x_2 \leq x_2$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3 - x_1)$$

- Fully subtractive: we subtract the smallest one to all the largest ones and we reorder with $x_1 \leq x_2 \leq x_3$

$$(x_1, x_2, x_3) \mapsto (x_1, x_2 - x_1, x_3 - x_1)$$

- Let $X \subset \mathbb{R}_+^d$ (one usually take $X = \mathbb{R}_+^d$) and let $(X_i)_{i \in I}$ be a finite or countable partition of X into measurable subsets.
- Let M_i be non-negative integer matrices so that $M_i X \subset X_i$.
- We define a *d*-dimensional continued fraction map over X as the map

$$F : X \to X \qquad F(x) = M_i^{-1} x \text{ if } x \in X_i$$

  We define $M(x) = M_i$ if $x \in X_i$.
- The associated continued fraction algorithm consists in iteratively applying the map F on a vector $x \in X$.
- This yields the sequence $(M(F^n(x)))_{n \geq 1}$ of matrices, called the continued fraction expansion of x.
- We then can interpret these matrices as incidence matrices of substitutions (with a choice that is highly non-canonical).

Jacobi-Perron substitutions

Consider for instance the Jacobi-Perron algorithm. Its projective version is defined on the unit square $(0,1)\times(0,1)$ by:

$$(\alpha,\beta)\mapsto\left(\frac{\beta}{\alpha}-\left\lfloor\frac{\beta}{\alpha}\right\rfloor,\frac{1}{\alpha}-\left\lfloor\frac{1}{\alpha}\right\rfloor\right)=\left(\left\{\frac{\beta}{\alpha}\right\},\left\{\frac{1}{\alpha}\right\}\right).$$

Its linear version is defined on the positive cone $X=\{(a,b,c)\in\mathbb{R}^3|0<a,b<c\}$ by:

$$(a,b,c)\mapsto(a_1,b_1,c_1)=(b-\lfloor b/a\rfloor a, c-\lfloor c/a\rfloor a, a).$$

Set $B=\lfloor b/a\rfloor a,\ C=\lfloor c/a\rfloor$. One has

$$\begin{pmatrix} a\\ b\\ c\end{pmatrix}=\begin{pmatrix}0&0&1\\1&0&B\\0&1&C\end{pmatrix}\begin{pmatrix}a_1\\ b_1\\ c_1\end{pmatrix}.$$

We associate with the above matrix the substitution

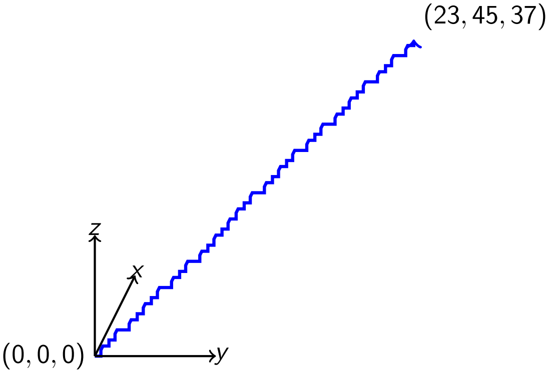
$$\sigma_{B,C}\colon 1\mapsto 2,\ 2\mapsto 3,\ 3\mapsto 12^B3^C$$

Applying Brun algorithm to $(23, 45, 37)$

Brun consists in subtracting the second largest entry to the largest

- Consider $a \leq b \leq c$
- Send (a, b, c) to $(a, b, c - b)$ and reorder

Applying Brun algorithm to $(23, 45, 37)$



S-adic expansions

One considers

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

S-adic expansions

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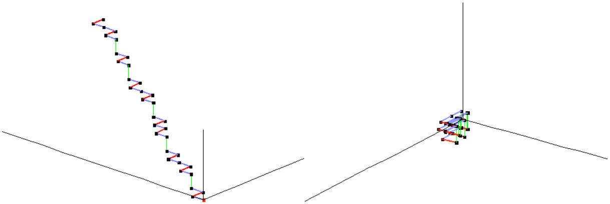
Convergence



Let p_k be the perfix of u of length k. Do the abelianizations of the p_k "converge" to the line ?
Convergence speed ? Type of convergence ? Weak ? strong ?

S-adic expansions

One considers

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

Combinatorially

- Frequencies with bounded remainders and balance

$$\exists C,\ \forall i \in \mathcal{A},\ \exists f(i) \text{ t.q. } \forall N\ |\text{Card}\{k \leq N,\ u_k = i\} - Nf(i)| \leq C$$

S-adic expansions

One considers

$$u = \lim_{n \to +\infty} \sigma_1 \sigma_2 \cdots \sigma_n(0)$$

Arithmetically

• Weak and strong convergence of multidimensional continued fraction algorithms

Theorem There exists $\delta > 0$ s.t. for almost every (α, β), there exists $n_0 = n_0(\alpha, \beta)$ s.t. for all $n \geq n_0$

$$|\alpha - p_n/q_n| < \frac{1}{q_n^{1+\delta}}$$

$$|\beta - r_n/q_n| < \frac{1}{q_n^{1+\delta}},$$

where p_n, q_n, r_n are given by Brun/Jacobi-Perron.

Brun [Ito-Fujita-Keane-Ohtsuki '93+'96]; Jacobi-Perron [Broise-Guivarc'h '99]

Lyapunov exponents for S-adic systems

- Let $\mathcal{S}$ be a finite set of unimodular substitutions

$\rightsquigarrow$ log-integrability

$$\int \log \max(\|A_1(\gamma)\|, \|A_1(\gamma)^{-1}\|) d\mu(\gamma) < \infty.$$

- Let (D, S, ν) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an ergodic subshift equipped with a probability measure ν

S is the shift acting on D
A subshift is a closed shift-invariant subset of sequences

- We consider the behaviour of the matrices $A_n(s) = M_{\sigma_0} \cdots M_{\sigma_n}$ for a generic $s = (\sigma_n) \in D$

Lyapunov exponents for S-adic systems

- Let $\mathcal{S}$ be a finite set of unimodular substitutions
- Let (D, S, ν) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an ergodic subshift equipped with a probability measure ν
- We consider the behaviour of the matrices $A_n(s) = M_{\sigma_0} \cdots M_{\sigma_n}$ for a generic $s = (\sigma_n) \in D$

The Lyapunov exponents $\theta_1, \theta_2, \ldots, \theta_d$ of (D, S, ν) are recursively defined by the ν-a.e. limit of

$$\theta_1 + \theta_2 + \cdots + \theta_k = \lim_{n\to\infty} \frac{1}{n} \log \| \wedge^k (M_{\sigma_0} \cdots M_{\sigma_{n-1}}) \|$$

where $\wedge^k$ denotes the k-fold wedge product

Lyapunov exponents for S-adic systems

- Let $\mathcal{S}$ be a finite set of unimodular substitutions
- Let (D, S, ν) with $D \subset \mathcal{S}^{\mathbb{N}}$ be an ergodic subshift equipped with a probability measure ν
- We consider the behaviour of the matrices $A_n(s) = M_{\sigma_0} \cdots M_{\sigma_n}$ for a generic $s = (\sigma_n) \in D$

The S-adic system (D, S, ν) satisfies the Pisot condition if

$$\theta_1 > 0 > \theta_2 \geq \theta_3 \geq \cdots \geq \theta_d$$

S-adic Pisot dynamics

Theorem [B.-Steiner-Thuswaldner]

- For almost every $(\alpha, \beta) \in [0,1]^2$, the S-adic system provided by the Brun multidimensional continued fraction algorithm applied to (α, β) is measurably conjugate to the translation by (α, β) on the torus $\mathbb{T}^2$
- For almost every Arnoux-Rauzy word, the associated S-adic system has discrete spectrum

Proof Based on

- "adic IFS" (Iterated Function System)
- Theorem [Avila-Delecroix]
  - The Arnoux-Rauzy S-adic system is Pisot
- Theorem [Avila-Hubert-Skripchenko]
  - A measure of maximal entropy for the Rauzy gasket
- Finite products of Brun/Arnoux-Rauzy substitutions have discrete spectrum [B.-Bourdon-Jolivet-Siegel]

The two-letter case [B.-Minervino-Steiner-Thuswaldner]

Let $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ be a primitive and algebraically irreducible sequence of unimodular substitutions over $\mathcal{A} = \{1, 2\}$

Assume that there is $C > 0$ such that

- for each $\ell \in \mathbb{N}$, there is $n \geq 1$ with

$$(\sigma_n, \ldots, \sigma_{n+\ell-1}) = (\sigma_0, \ldots, \sigma_{\ell-1}) \quad \text{recurrence}$$

- the language $\mathcal{L}_{\boldsymbol{\sigma}}^{(n+\ell)}$ has bounded discrepancy with the same bound C

Then the S-adic shift $X_{\boldsymbol{\sigma}}$ has pure discrete spectrum

Pisot adic dynamics

- Substitutions produce hierarchical ordered structures (infinite words, point sets, tilings) that display strong self-similarity properties
- Substitutions are closely related to induction (first return maps, Rokhlin towers, renormalization etc.)
- We consider substitutions that create a hierarchical structure with a significant amount of long range order
- And we go beyond algebraicity via the S-adic formalism