Interval exchange transformations
Part II: Minimality, Rauzy induction and Teichmüller flow)

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Rotations

Recall that for rotation we have:

Theorem

Let α be irrational, and X_α be the Sturmian shift associated to the rotation T_α. Then:

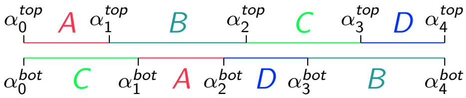
- $p_{X_\alpha}(n) = n + 1$, *in particular X_α has* **0 entropy***;*
- *the shift X_α is* **minimal** *(all orbits are dense);*
- *(Hecke (1922), Ostrowski (1922)) any clopen $Y \subset X_\alpha$ has bounded remainder: there exists μ_Y and C_Y so that*

$$\forall x \in X_\alpha, \forall n \geq 0, \quad \left| \sum_{k=0}^{n} (\chi_Y(T_\alpha^n x) - \mu_Y) \right| \leq C_Y.$$

In particular, the shift X_α is **uniquely ergodic***.*

Interval exchange transformations

An interval exchange transformation is a piecwise translation of the interval that is a bijection from $I \backslash \{\alpha_1^{top}, \ldots, \alpha_{d-1}^{top}\}$ to $I \backslash \{\alpha_1^{bot}, \ldots, \alpha_{d-1}^{bot}\}$.



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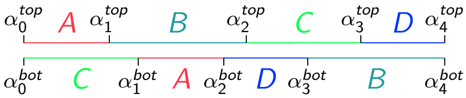


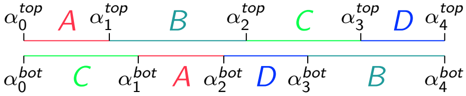
The above interval exchange can be defined from:

- a "permutation" $\pi = \begin{pmatrix} A & B & C & D \\ C & A & D & B \end{pmatrix}$,
- a length vector $\lambda = (\lambda_A, \lambda_B, \lambda_C, \lambda_D)$.

Main motivation: rational billiards

(. . . Sage . . .)

Coding



As we did for rotations, we could code orbits in $\{A, B, C, D\}^{\mathbb{Z}}$ (except the singular ones). We obtain a shift $X_{\pi,\lambda}$ and a factor map $p : X_{\pi,\lambda} \to I$.

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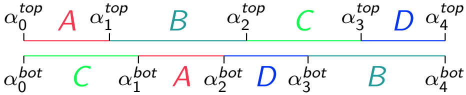

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All orbits of the iet $T_{\pi,\lambda}$ has one preimage in $X_{\pi,\lambda}$ except the singular ones that have two (i.e. the future orbits of $\alpha_1^{bot}, \ldots, \alpha_2^{bot}$ and the past orbits of $\alpha_1^{top}, \ldots, \alpha_{d-1}^{bot}$).

Connections

$\Sigma^{top} := \{\alpha_1^{top}, \ldots, \alpha_{d-1}^{top}\}$ (singularities of T)
$\Sigma^{bot} := \{\alpha_1^{bot}, \ldots, \alpha_{d-1}^{bot}\}$ (singularities of T^{-1}).
A *connection* is a triple (m, α, β) where $m \geq 0$, $\alpha \in \Sigma^{top}$, $\beta \in \Sigma^{bot}$ and $T^m\beta = \alpha$.

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- If the length data λ is rational

$$\dim_{\mathbb{Q}} \left(\mathbb{Q}\frac{\lambda_1}{\lambda_d} + \mathbb{Q}\frac{\lambda_2}{\lambda_d} + \ldots + \mathbb{Q}\frac{\lambda_{d-1}}{\lambda_d} \right) = 1$$

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- If the length data λ is maximally irrational

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  then there are no connection.

Coding

Theorem

Let $X_{\pi,\lambda}$ be the shift associated to an interval exchange transformation $T_{\pi,\lambda}$ on d intervals with π irreducible. Then the following are equivalent

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In general

$$\lim_{n\to\infty} \frac{p_{\pi,\lambda}(n)}{n} = (d-1) - \#\mathit{connections}.$$

Minimality: Keane theorem

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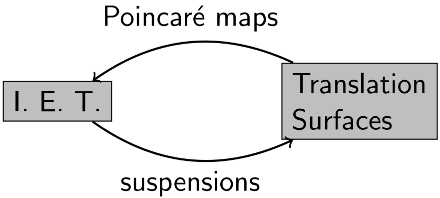
Corollary

In a rational billiard, excepted countably many directions the flow is minimal.

Back and forth between iet and translation surfaces



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