Rigidity results in cellular automata theory: probabilistic and ergodic approach

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Outline

Symbolic Dynamics (in dimension 1)

- Consider A a finite set and $X = A^{\mathbb{Z}}$ the set of two-sided sequences

$$\mathbf{x} = (x_i)_{i\in\mathbb{Z}} = (\ldots x_{-i}\ldots x_0\ldots x_i\ldots)$$

of symbols in A. Analogously one defines $X = A^{\mathbb{N}}$ the set of one-sided sequences in A. Both are called full-shifts. For simplicity we restrict to the two-sided case.

- The space X is compact for the product topology and metrizable (two points are close if they coincide near the origin).

- A natural dynamical system is the shift map $\sigma : X \to X$, where $\sigma(\mathbf{x}) = (x_{i+1})_{i\in\mathbb{Z}}$. It is a homeomorphism of X.

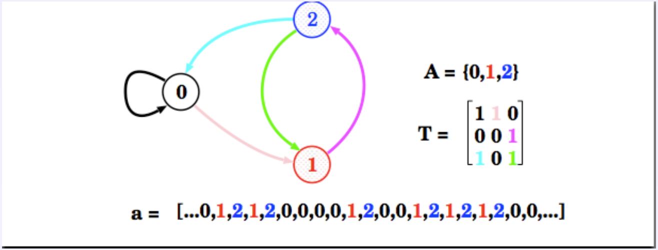
- Subshifts: if $Y \subset X$ is closed and $\sigma(Y) \subset Y$ it is called a subshift. Consider the orbit closure of points in X as a first example.

Subshifts of Finite Type (SFT)

Special subshifts are subshifts of finite type; they look like Markov chains in probability theory. Y is a subshift of finite type if there is a finite subset $\mathcal{W}$ of words in A of a given length L such that for any $\mathbf{y} \in Y$ and $i \in \mathbb{Z}$,

$$y_i \dots y_{i+L-1} \notin \mathcal{W}$$

Example: $A = \{0, 1, 2\}$ and $\mathcal{W} = \{02, 10, 11, 22\}$:



Block maps

- A second kind of important dynamics are given by continuous and shift commuting maps of a subshift Y: $F : Y \to Y$. That is: F is continuous and $F \circ \sigma = \sigma \circ F$.

- They are called block maps since there is a local map,

$$f : A^{m+a+1} \to A$$

$a, m \in \mathbb{N}$ ($a =$ anticipation and $m =$ memory respectively), such that $\forall\ i \in \mathbb{Z}, \forall\ \mathbf{y} \in Y$

$$F(\mathbf{y})_i = f(y_{i-m}, \ldots, y_{i+a})$$

- Cellular automaton: Y is a mixing shift of finite type (i.e., two words in Y can be glued in a very strong way inside Y), typically the fullshift

Main questions and **evidence !!!**

Randomization evidence (here a CA on $\{0,1,2\}^{\mathbb{Z}}$):

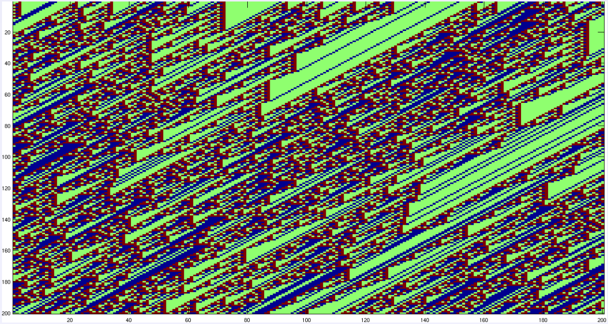


Figure: Iteration of a CA

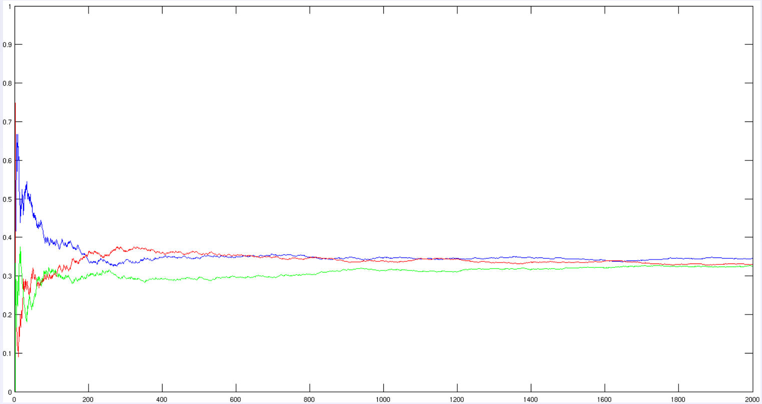


Figure: Frequency of symbols after "Cesàro mean"

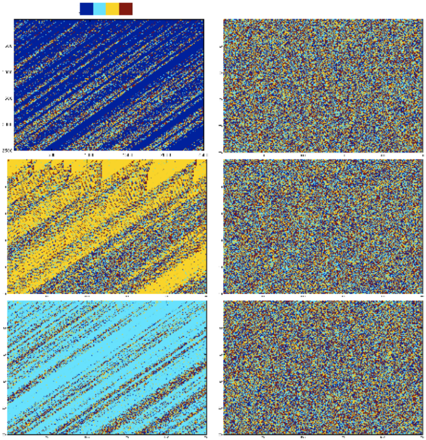


Figure: Other Automata, same phenomena

Recall

Entropy: Classical measure of complexity of the dynamics with respect to an invariant measure μ

$$h_\mu(\sigma) = -\lim_{N\to\infty} \frac{1}{N} \sum_{a_0,\ldots,a_{N-1}} \mu([a_0 \ldots a_{N-1}]) \log \mu([a_0 \ldots a_{N-1}])$$

where $[a_0 \ldots a_{N-1}] = \{\mathbf{y} \in Y \ : \ y_0 \ldots y_{N-1} = a_0 \ldots a_{N-1}\}$.

A measure of maximal entropy (for the shift map here) is one for which:

$$h_\mu(\sigma) = \sup_\nu h_\nu(\sigma)$$

Let $F : Y \to Y$ be a surjective or onto block map of a mixing subshift of finite type or cellular automaton.

Question 1: Given a shift invariant probability measure μ on Y describe if it exists the limit of the sequence $(F^n\mu \ : \ n \in \mathbb{N})$. Every weak limit of a subsequence is invariant for F (and the shift). It is also interested the convergence when $N \to \infty$ of the Cesàro mean

$$\mathcal{M}_{\mu}^{N}(F) = \frac{1}{N} \sum_{n=0}^{N-1} F^n\mu$$

One says F asymptotically randomizes μ if the limit of the Cesàro mean converges to the maximal entropy measure.

Question 2: Study invariant measures of F and for the joint action of F and σ: i.e. probability measures μ such that for any Borel set $B \in \mathcal{B}(X)$ and $n \in \mathbb{N}, m \in \mathbb{Z}$

$$F^n\mu(B) := \mu(F^{-n}B) = \mu(B)$$

or

$$F^n \circ \sigma^m\mu(B) := \mu(F^{-n} \circ \sigma^{-m}B) = \mu(B)$$

– A natural invariant measure for F is the maximal entropy one for the shift map. In fact F is onto if and only if the maximal entropy measure is F-invariant (Coven-Paul).

– Depending on the subshift Y and dynamical properties of F it is possible to construct other invariant measures; nevertheless in some cases strong rigidities appear (for example when strong forms of expansivity exist) .

Looking for a good class of examples:

Dichotomy:

– From Glasner and Weiss result in topological dynamics one gets essentially that either the map F is almost equicontinuous or sensitive to initial conditions, and in the last class most interesting known examples (and in fact comes from Nasu's reductions) are expansive or positively expansive maps.

– In the equicontinuous case or systems with equicontinuous points, orbits tend to be periodic and invariant measures can be more or less described but are not nice.

– If the maps are positively expansive they are conjugate with shifts of finite type (M-Blanchard, Nasu, M-Boyle), so we have two commuting shifts of finite type with the same maximal entropy measure. In this last case there can still exists an equicontinuous direction so invariant measures are as in previous cases.

– Good examples: (positively) expansive maps without equicontinuous directions; even if not easy to know a priori how they are constructed, there are some advances by Boyle-Lind and Mike Hochman from the point of view of expansive sudynamics. Main classes with this features correspond to algebraic maps.

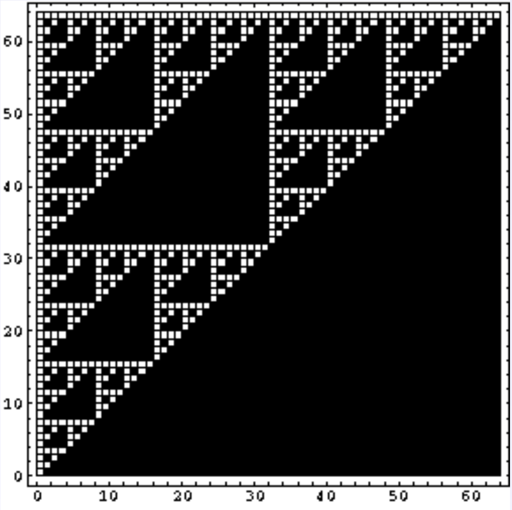
Basic Example: addition modulo 2 or *Ledrappier's three dot problem*

Let $X = \{0,1\}^{\mathbb{Z}}$ (see X as an Abelian group with coordinatewise addition modulo 2) and $F : X \to X$ given by $F(x) = id \; + \; \sigma$, where σ is the shift map in X. That is, $F(\mathbf{x})_i = x_i + x_{i+1}$. It is a 2-to-1 onto map.

– **In relation to Question 2:** Natural invariant measures are the uniform Bernoulli measure $\lambda = (1/2, 1/2)^{\mathbb{Z}}$ and measures supported on periodic orbits of F. But there exist other invariant measures of algebraic origin that has been described in works by M. Einsiedler, E. Lindenstrauss, B. Kitchens, K. Schmidt.

– **In relation to Question 1:** In general the limit does not exist: Pascal triangle modulo 2 in Bernoulli case (we only draw one-sided sequences).

x	=	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	...	...	x_{2n}
F(x)	=	x_0+x_1	x_1+x_2	x_2+x_3	x_3+x_4	x_4+x_5	x_5+x_6	x_6+x_7	...	...	...	...
F^2(x)	=	x_0+x_2	x_1+x_3	x_2+x_4	x_3+x_5	x_4+x_6	x_5+x_7	...	...	...	...	...
F^3(x)	=	$x_0+x_1+x_2+x_3$	$x_1+x_2+x_3+x_4$	$x_2+x_3+x_4+x_5$	$x_3+x_4+x_5+x_6$	...	...	...	...	...	...	...
F^4(x)	=	x_0+x_4	x_1+x_5	x_2+x_6	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
...	...	...	...	...	...	...	...	...	...	...	...	...
F^{2n-1}(x)	=	$x_0+\ldots+x_{2n-1}$	...	...	...	...	...	...	...	...	...	...
F^{2n}(x)	=	x_0+x_{2n}	...	...	...	...	...	...	...	...	...	...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— Assume $\mu = (\pi_0, \pi_1)^{\mathbb{Z}}$ be a Bernoulli non-uniform measure on X with $\pi_0 = \mu(x_i = 0)$, $\pi_1 = \mu(x_i = 1)$.

— A simple induction yields to:

$$\mu\left(\sum_{i \in I} x_i = a\right) = \frac{1}{2}\left(1 + (-1)^a (\pi_0 - \pi_1)^{\#I}\right)$$

— Thus,

$$F^n\mu[a]_0 = \mu\left(\sum_{k \in I(n)} x_k = a\right) = \frac{1}{2}\left(1 + (-1)^a (\pi_0 - \pi_1)^{\#I(n)}\right)$$

where $I(n) = \{0 \leq k \leq n \ : \ C_n^k = 1 \bmod 2\}$.

— If $a = 0$ for $n = 2^m$ the limit is $\pi_0^2 + \pi_1^2$ and for $n = 2^m - 1$ the limit is $\frac{1}{2}$.

- But the Cesàro mean converges:

$$\mathcal{M}_\mu^N(F)[a]_0 = \frac{1}{N}\sum_{n=0}^{N-1} F^n\mu[a]_0 = \frac{1}{2} + \frac{(-1)^a}{2}\frac{1}{N}\sum_{n=0}^{N-1}(\pi_0 - \pi_1)^{\#I(n)}$$

since $\lim_{N\to\infty} \frac{\{0\leq n<N \ : \ \#I(n)\geq \alpha \log\log N\}}{N} = 1$ for some $\alpha \in (0, 1/2)$ (Lucas' lemma)

then the limit is $\frac{1}{2}$ and for ℓ coordinates is $\frac{1}{2^\ell}$. This was observed by D. Lind in 84 for $F = \sigma^{-1} + \sigma$.

This result reinforce the idea that the uniform Bernoulli measure $\lambda = (1/2, 1/2)^{\mathbb{Z}}$ must be the unique invariant measure of (F, σ) verifying "some conditions to be determined".

• **Question 3:** Find conditions to ensure the maximal entropy measure is the unique solution to Questions 1 and 2.

— In relation with Question 1 there are two points of view. One is to consider measures μ of increasing complexity in correlations: Markov, Gibbs, other chain connected measures; represent them as "independent processes" and prove that the limit of the Cesàro mean converges to λ. The other is motivated in harmonic analysis and Lind's work; the idea is to define a class of mixing measures such that the Cesàro mean of any of them converges.

— In relation with Question 2 the type of solutions looks like the $(\times 2, \times 3)$-Furstenberg problem in $\mathbb{R}/\mathbb{Z}$: F (or σ) ergodic and σ (resp. F) with positive entropy for the invariant measure. While ergodicity of one transformation can be changed for a weaker condition the positivity of entropy cannot be dropped for the moment. Proofs strongly rely on entropy formulas. These conditions already appear in Rudolph's solution to $(\times 2, \times 3)$ problem and all recent improvements and related results by Host, Lindenstrauss, Einsiedler, ...

Iteration of measures: harmonic analysis point of view

— Lind 84, Pivato-Yassawi 02, 04, Host-M-Martínez 03, M-Martínez-Pivato-Yassawi 06

— Let $(A, +)$ be a finite Abelian group.

— $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ block map with local block map f where

$$f(x_{i-m} \ldots x_{i+a}) = \sum_{k=-m}^{a} f_k(x_{i+k})$$

— a character $\chi : A^{\mathbb{Z}} \to \mathbb{T}^1$ in $\widehat{A}^{\mathbb{Z}}$ is given by $\chi = \bigoplus_{k \in \mathbb{Z}} \chi_k$ where χ_k are characters of A and $\chi_k = 1$ for all but finitely many terms in this product.

— the rank(χ) is the $\#$ of non trivial characters χ_k in $\bigoplus_{k \in \mathbb{Z}} \chi_k$.

— the Haar or uniform Bernoulli measure λ is characterized by

$$\lambda(\chi) = \int_{A^{\mathbb{Z}}} \chi d\lambda = 0 \quad \forall \chi \neq 1.$$

Definition (Pivato-Yassawi 02)

μ is harmonically mixing if $\forall \varepsilon > 0 \; \exists N(\varepsilon) > 0$ such that $\forall \chi \in \widehat{A}^{\mathbb{Z}}$:

$$\text{rank}(\chi) > N(\varepsilon) \Rightarrow |\mu(\chi)| < \varepsilon$$

If $A = \mathbb{Z}_p$, then a Markovian measure with strictly positive transitions is harmonically mixing.

Definition (Pivato-Yassawi 02)

- The block map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is diffusive if

$$\forall \chi \not\equiv 1 : \lim_{n\to\infty} \text{rank}\left[\chi \circ F^n\right] = \infty$$

- F is diffusive in density if $\exists J \subseteq \mathbb{N}$ of density 1 s.t.

$$\lim_{\substack{n\to\infty \\ n\in J}} \text{rank}\left[\chi \circ F^n\right] = \infty.$$

Theorem (Pivato,Yassawi 02,04; Ferrari,M,Martínez,Ney 00)

Let A be a finite abelian group. Then if the f_k, $k = -m, \ldots, a$, are commuting automorphisms of A and at least two are nontrivial, then F is diffusive in density. Therefore for any harmonically mixing measure μ:

$$\mathcal{M}_\mu(F) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} F^n \mu = \lambda$$

Iteration of measures: regeneration of measures point of view

— Ferrari,M,Martínez,Ney 2000, Host,M,Martínez 2003, M,Martínez,Sobottka 2006

— Let μ be any shift invariant probability measure on $A^{\mathbb{Z}}$ and consider $w = (\ldots, w_{-2}, w_{-1}) \in A^{-\mathbb{N}}$. Denote by μ_w the conditional probability measure on $A^{\mathbb{N}}$.

— One says that μ has *complete connections* if given $a \in A$ and $w \in A^{-\mathbb{N}}$, $\mu_w([a]_0) > 0$. If μ is a probability measure with complete connections, one define the quantities γ_m, for $m \geq 1$, by

$$\gamma_m = \sup \left(\left| \frac{\mu_v([a]_0)}{\mu_w([a]_0)} - 1 \right| \ : \ v, w \in A^{-\mathbb{N}}; \ v_{-i} = w_{-i}, \ 1 \leq i \leq m \right)$$

— If $\sum_{m \geq 1} \gamma_m < \infty$ one says μ has *summable decay of correlations*.

Theorem (Ferrari-M-Martínez,Ney 00)

Let $(A, +)$ be a finite Abelian group, μ a probability measure on $A^{\mathbb{Z}}$ with complete connections and summable decay of correlations. Let $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ as before. Then $\forall\ w \in A^{-\mathbb{N}}$ $\exists$ Cesàro mean distribution $\mathcal{M}_{\mu_w}(F) = \lambda$.

Several generalizations by: M,Martínez,Pivato,Yassawi 06, M,Martínez,Sobottka 05.

Main Ideas: regeneration of measures.

— Let $(T_i : i \geq 1)$ be an increasing sequence of non-negative integer random variables. For every finite subset L of $\mathbb{N}$ let

$$\mathbf{N}(L) = \left|\{i \geq 1 : T_i \in L\}\right| .$$

One says that $(T_i : i \geq 1)$ is a stationary renewal process with finite mean interrenewal time if
(1) $(T_i - T_{i-1} : i \geq 2)$ are independent identically distributed with finite expectation, they are independent of T_1 and $\mathbb{P}(T_2 - T_1 > 0) > 0$.

(2) For $n \in \mathbb{N}$, $\mathbb{P}(T_1 = n) = \frac{1}{\mathbb{E}(T_2 - T_1)} \mathbb{P}(T_2 - T_1 > n)$.

The above conditions imply the stationary property: for every finite subset L of $\mathbb{N}$ and every $a \in \mathbb{N}$ the random variables $\mathbf{N}(L)$ and $\mathbf{N}(L + a)$ have the same distribution.

Theorem (Ferrari-M-Martínez-Ney)

Let μ be a shift invariant probability measure on $A^{\mathbb{Z}}$ with complete connections and summable decay of correlations. There exists a stationary renewal process $(T_i : i \geq 1)$ with finite mean interrenewal time such that for every $w \in A^{-\mathbb{N}}$, there exists a random sequence $z = (z_i : i \geq 1)$ with values in A and distribution μ_w such that $(z_{T_i} : i \geq 1)$ are i.i.d. uniformly distributed in A and independent of $(z_i : i \in \mathbb{N} \setminus \{T_1, T_2, \ldots\})$.

— From the construction of the renewal process in [FMMN] one also gets the following properties:

(1) There exists a function $\rho : \mathbb{N} \to \mathbb{R}$ decreasing to zero such that $\mathbb{P}(\mathbf{N}(L) = 0) \leq \rho(|L|)$, for any finite subset L of $\mathbb{N}$.

(2) Given $n, \ell \in \mathbb{N}^*$, $1 \leq k_1 < ... < k_\ell \leq n$ and $j_1, ..., j_l \in \mathbb{N}$, for all $a_1, ..., a_n \in A$.

$$\mu_w \left(z_i = a_i, i \in \{1, ..., n\}; T_{j_1} = k_1, ..., T_{j_\ell} = k_\ell\right) =$$

$$\frac{1}{|A|^\ell} \mu_w \left(z_i = a_i, i \in \{1, ..., n\} \setminus \{k_1, ..., k_\ell\}; T_{j_1} = k_1, ..., T_{j_\ell} = k_\ell\right)$$

(3) For any $n \in \mathbb{N}$ and $v \in A^*$, $\mu_w \left(\{\mathbf{N}(\{0, \ldots, n-1\}) > 0\} \cap [v]_n\right)$ does not depend on $w \in A^{-\mathbb{N}}$;

Theorem (Host,M,Martínez 03; use ideas in (FMMN))

A shift invariant probability measure with complete connections and summable decay of correlations is harmonically mixing.

Results on (F, σ)-invariant measures: the basic theorem in the theory concerns our basic example.

Theorem (Basic Theorem: Host-M-Martínez)

Let $F : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$, $F = id + \sigma$. If μ is (F, σ) invariant with $h_\mu(F) > 0$ and σ-ergodic, then $\mu = \lambda$.

• **Idea of proof:** consider μ a (F, σ) invariant measure.

1) Let $\mathcal{B}_1 = F^{-1}\mathcal{B}$ and for a.e. $x \in X$ consider $\mu_x(\cdot) = \mathbb{E}(\cdot|\mathcal{B}_1)$. It is concentrated on $\{x, x + \mathbb{1}\}$, where $\mathbb{1} = \ldots 1111111 \ldots$.

2) Define $\phi(x) = \mu_x(\{x + \mathbb{1}\})$. Then

$$\phi \circ \sigma(x) = \phi(x)$$

3) **Ergodicity** of μ for σ implies ϕ constant μ-a.e., so $F\mu$-a.e., which implies

$$\phi \circ F = \phi \circ \sigma = \phi, \ \mu - \text{a.e.} \qquad (*)$$

4) Define $E = \{x \in X \ : \ \phi(x) > 0\}$ and prove that

$$\mu_x(\{x\}) = \mu_x(\{x + \mathbb{1}\}) = \frac{1}{2}$$

for μ-a.e. x in E

5) E is σ-invariant by (*), then by **ergodicity** $\mu(E) = 0 \vee 1$.

6) Entropy formula:

– Let $\alpha = \{[0]_0, [1]_0\}$,

$$h_\mu(F) = H_\mu(\alpha \mid \mathcal{B}_1)$$

– Observe that when $x \in [a]_0$ then $\mu_x([a]_0) = \mu(\{x\})$ since $x + \mathbb{1} \notin [a]_0$ for $a = 0, 1$. Then

$$h_\mu(F) = -\int_X \log(\mu_x(\{x\}))d\mu(x)$$

– If $h_\mu(F) > 0$ then $\mu(E) > 0$. From **ergodicity**, $\mu(E) = 1$;

– $\mu_x(\{x\}) = \mu_x(\{x + \mathbb{1}\}) = \frac{1}{2}$ for μ-a.e. $x \in X$;

– $h_\mu(F) = \log(2)$, thus $\mu = \lambda$ that is the unique maximal entropy measure for F.

Some generalizations:

Theorem (Host-M-Martínez)

Let $F : \mathbb{Z}_p^{\mathbb{Z}} \to \mathbb{Z}_p^{\mathbb{Z}}$ be linear. Let μ be (F, σ)-invariant. If $h_\mu(F) > 0$ and μ is ergodic for σ then μ is the uniform Bernoulli measure.

Theorem (Host-M-Martínez)

Let $F : \mathbb{Z}_p^{\mathbb{Z}} \to \mathbb{Z}_p^{\mathbb{Z}}$ be linear. Let μ be (F, σ)-invariant. If $h_\mu(F) > 0$, μ is ergodic for (σ,F) and $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{p(p-1)})$, then μ is the uniform Bernoulli measure.

– A block map $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is algebraic if $A^{\mathbb{Z}}$ is a compact abelian topological group and F and the shifts are endomorphisms of such group.

Theorem (Pivato)

Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an algebraic bipermutative block map. Then, if μ is totally ergodic for σ, $h_\mu(F) > 0$ and $Ker(F)$ has no shift invariant subgroups, then μ is the Haar measure.

Theorem (Einsiedler)

Let α be an algebraic $\mathbb{Z}^d$-action of a compact 0-dimensional abelian group, and some additional algebraic conditions. Positive entropy in one direction and totally ergodicity of the action imply Haar measure.

Theorem (Sablik)

Let $F : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be an algebraic bipermutative block map and Σ be a (F,σ)-invariant closed subgroup of $A^{\mathbb{Z}}$. Fix $k \in \mathbb{N}$ such that any prime divisor of $|A|$ divides k. If μ is (F,σ)-invariant with supp$(\mu) \subseteq \Sigma$ such that:
— μ is ergodic for (σ, F),
— $h_\mu(F) > 0$,
— $\mathcal{I}_\mu(\sigma) = \mathcal{I}_\mu(\sigma^{kp_1})$, where p_1 is the smallest common period of the elements in Ker(F),
— any finite shift invariant subgroup of $\cup_{n\in\mathbb{N}} Ker(F^n) \cap \Sigma$ is dense in Σ,
then μ is the Haar measure of Σ.

— **Remark.** From last theorems it is possible to deduce the same kind of results for some classes of positively expansive and expansive block maps of a fullshift, *a priori* not algebraic.

Final Comments:

— Change "complete connections and summable decay of correlations" by some mixing property for the shift map.

— The asymptotic randomization does not require full support of initial measure and positive entropy w.r.t. the shift map: there exist shift invariant measures μ on $\{0, 1\}$ with $h_\mu(\sigma) = 0$ that are asymptotically randomized by $F = id + \sigma$ (Pivato-Yassawi examples 2006).

— **Question:** Cesàro means exist for expansive and positively expansive block maps of a mixing shift of finite type ?; how the limit is related with the unique maximal entropy measure ?
Partial results for classes of right permutative cellular automata: with associative local rules, or N-scaling local rules (Host,M,Martínez); they can be seen as the product of an algebraic CA with a shift: here measures are not asymptotically randomized but the limit are the product of a maximal measure with a periodic measure.

GRACIAS !!!